CONTROLLABILITY CRITERION FOR ONE LINEAR MATRIX DELAYED EQUATION

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Abstract: In this paper fundamental matrix of solutions of differential linear matrix equations with delay was presented. Necessary and sufficient condition for controllability of correspondent control problem was defined and control was built. Paper contains calculated examples.

Keywords: fundamental matrix of solutions, controllability, differential delayed equation, matrix exponential

1 INTRODUCTION

Many researches are devoted for solving problems related to differential equations with delay, such as, [1]-[8].
In many dynamic systems a delay appears. For example, in the simplest electronical circuits, the delay effect appears in voltage and current signals because of elements such as capacitors and inductors, respectively. Such dynamic systems can be described by systems of differential equations with aftereffects.
This paper is devoted to computing the solution of differential linear matrix equation with delay, described as follows, $\dot{x}(t) = Ax(t) + Ax(t-\tau)$. To find the fundamental matrix of solutions the “step by step method” has been used. The solution has been presented with help of the special matrix function - matrix exponential. Matrix exponential was used for solving differential equations by Krasovsky [9], [10] and for solving systems with aftereffects by many authors, e.g. Boichuk, Diblík, Khusainov, Růžičková, Shuklin [5] - [8].
The corresponding control problem has been built, a necessary and sufficient condition for controllability has been proposed and the control has been built.

2 LINEAR MATRIX EQUATION WITH DELAY

Let us consider the following Cauchy problem

$$\dot{x}(x) = Ax(t) + Ax(t-\tau) + f(t), \quad x(t) = \varphi(t), \quad -\tau \le t \le 0,$$

here $x(t) = (x_1(t), x_2(t), .., x_n(t))^T$ is vector of system state, $f(t) = (f_1(t), f_2(t), .., f_n(t))^T$ is known function of disturbance, A is square matrix, $\tau > 0, \tau \in R$ is a constant delay.

To solve this Cauchy problem let us find the fundamental matrix of solution of this equation. Fundamental matrix would be a solution of following matrix equation

$$\dot{X}(t) = AX(t) + AX(t-\tau), \tag{1}$$

with initial condition

$$X(t) = I, \qquad -\tau \le t \le 0,$$

where A is square matrix, I is identity matrix, $\tau > 0, \tau \in R$ is a constant delay.

Definition 2.1 *Let A be a square matrix. Matrix exponential is defined by*

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!},$$

where I is the identity matrix.

Theorem 2.2 *The solution of equation* (1) *with identity initial condition has the recurrent form:*

$$X_{n+1}(t) = e^{A(t-n\tau)}X_n(n\tau) + \int_{n\tau}^{t} e^{A(t-s)}AX_n(s-\tau)ds,$$

where $X_n(t)$ is defined on the interval $(n-1)\tau \le t \le n\tau$.

Theorem 2.3 *The fundamental matrix of solutions of equation* (1) *has the form:*

$$\mathrm{X}_0 = \begin{cases} \Theta, & -\infty \le t < -\tau \\ I, & -\tau \le t < 0 \\ 2e^{At} - I, & 0 \le t \le \tau \\ 2e^{At} + 2e^{A(t-\tau)}\left(A(t-\tau) - I\right) + I, & \tau \le t \le 2\tau \\ \ldots & \\ \sum\limits_{m=0}^{k-1} 2e^{A(t-m\tau)} \sum\limits_{p=0}^{m} (-1)^{p+m} A^p \frac{(t-m\tau)^p}{p!} + (-I)^k, & (k-1)\tau \le t < k\tau, \quad k \in Z^+, \end{cases}$$

where Θ is zero matrix, I is identity matrix.

Let we have the linear heterogeneous equation with delay

$$\dot{x}(t) = Ax(t) + Ax(t-\tau) + f(t). \tag{2}$$

If we have initial condition in the form

$$x(t) = \varphi(t), \quad -\tau \le t \le 0, \tag{3}$$

where $\varphi(t) \in C^1[-\tau, 0]$, then we could write the following result.

Theorem 2.4 *The solution of heterogeneous equation* (2) *with the initial condition* (3) *has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_{0}^{t} \mathrm{X}_0(t-\tau-s)f(s)ds,$$

where $\mathrm{X}_0(t)$ is fundamental matrix of solutions of the matrix equation (1), *defined in Theorem 2.3.*

3 CONTROLLABILITY OF THE LINEAR MATRIX SYSTEM WITH DELAY

3.1 General terms

Let X is the space of states of dynamic system; U is the set of the controlled effects (controls). Let $x = x(x_0, u, t)$ is the vector that characterizes state of the dynamic system in moment of time t, by the initial condition x_0, $x_0 \in X$, $(x_0 = x|_{t=t_0})$ and by the control function u, $u \in U$.

Definition 3.1 *The state x_0 is called controllable state in the class U (controlled state), if there are exist control $u(x_0) \in U$ and the number T, $t_0 \leq T$ such that $x(x_0, u(x_0), T) = 0$.*

Definition 3.2 *If every state $x_0 \in X$ of the dynamic system is controllable, then we say that the system is controllable (controlled system).*

3.2 Construction of control for system with delay

Consider the following Cauchy's problem:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Ax(t-\tau) + Bu(t), t \geq 0, \\ x(0) = x_0, \ \ x(t) = \varphi(t), \ \ -\tau \leq t < 0, \end{array} \tag{4}$$

where $x = (x_1, ..., x_n)^T$ is the vector of phase coordinates, $x \in X$, $u(t) = (u_1(t), ..., u_r(t))^T$ is the control function, $u \in U$, U is the set of piecewise-continuous functions; A, B are constant matrixes of dimensions $(n \times n)$, $(n \times r)$ respectively, τ is the constant delay.

Theorem 3.3 *For controllability of linear system with delay (4) is necessary and sufficient to next condition to hold: $t \geq (k-1)\tau$ and $rank(S) = n$, where*

$$S = \{B\ (AB)\ (A^2B)\ ...\ (A^{n-1}B)\},$$

hence S is a matrix which was achieved by recording matrixes B, AB,..., $A^{n-1}B$ side by side.

Theorem 3.4 *Let $t_1 \geq (k-1)\tau$ and the necessary and sufficient condition for controllability is implemented:*

$$rank(S) = rank\left(\{B\ (AB)\ (A^2B)\ ...\ (A^{n-1}B)\}\right) = n.$$

Then the control function can be taken as

$$u(t) = [\mathrm{X}_0(t_1 - \tau - t)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1 - \tau - s)BB^T[\mathrm{X}_0(t_1 - \tau - s)]^T ds \right]^{-1} \mu, \tag{5}$$

where

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s) ds.$$

4 EXAMPLES

Let us consider few examples of controllability researches of the linear matrix systems with delay.

Example 4.1

Let us have the differential equation of 3-th degree with a constant delay:
$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t)$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

As we see $\tau = 1, n = 3$. We want to know if this system is controllable so let us check the necessary and sufficient condition. We will find the matrix S:

$$S_1 = \{B\ (AB)\ (A^2B)\} = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 3 & 3 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have, $rank(S_1) = 2$, so the system is not controllable.

Example 4.2

Let us have the differential equation of 3-th degree with a constant delay:

$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t)$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$.

As we see $\tau = 1, n = 3$. We want to know if this system is controllable so let us check the necessary and sufficient condition. We will find the matrix S:

$$S_2 = \{B\ (AB)\ (A^2B)\} = \begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

We have, $rank(S_2) = 3$, so the system is controllable.
Let us construct such control function, that move system in time moment $t_1 = 2$ in point $x_1 = (1,1,1)^T$, using initial condition $x_0(t) = \varphi(t) = (0,0,0)^T$, $-1 \le t \le 0$. Using the result of the theorem (3.4) we write:

$$u(t) = [\mathrm{X}_0(t_1 - \tau - t)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1 - \tau - s)BB^T [\mathrm{X}_0(t_1 - \tau - s)]^T ds \right]^{-1} \mu,$$

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s)ds.$$

While $\varphi(t) = (0,0,0)^T$, $-1 \le t \le 0$ then $\mu = (1,1,1)^T$. So, we have

$$u(t) = [\mathrm{X}_0(1-t)B]^T \left[\int_0^{2} \mathrm{X}_0(1-s)BB^T [\mathrm{X}_0(1-s)]^T ds \right]^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

While $t_1 = 2$, then $k = 2$ and, using (5) we can calculate

$$u(t) = \begin{cases} \left(0.24e^t - 0.12, \quad (0.12t^2 - 1.7t + 2.45)e^t - 0.86\right)^T, & 0 \le t < 1 \\ \left((0.089t + 0.064)e^t + 0.12, \quad (0.05t^3 - 0.59t^2 + 0.22t - 0.17)e^t + 0.86\right)^T, & 1 \le t < 2 \end{cases}$$

5 CONCLUSION

In this paper a fundamental matrix of solutions of the one delayed differential equation was built. The necessary and sufficient condition for controllability for corresponding control problem was defined and control was built. Two examples were given to illustrate the proposed theory.

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