STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: Stochastic differential equations used to describe physical phenomena, which are also subject to random influences. Solution of the stochastic model is a random process. In the presented contribution the stochastic differential equation is defined and its basic properties are listed.

Keywords: stochastic differential equation, white noise, Brownian motion, Wiener process

1 INTRODUCTION

Definition 1. If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \setminus \mathcal{F}$ is the complement of $\mathcal{F}$ in Ω

(iii) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.
A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \longrightarrow [0,1]$ such that

(a) $P(\emptyset) = 0, P(\Omega) = 1$.

(b) if $A_1, A_2, \dots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space. It is called a complete probability space if $\mathcal{F}$ contains all subsets G of Ω with P-outer measure zero, i.e. with

$$P^*(G) := \inf\{P(F); F \in \mathcal{F}, G \subset F\} = 0.$$

2 BROWNIAN MOTION

One of the simplest continuous-time stochastic processes is Brownian motion. This was first observed by botanist Robert Brown. He observed that pollen grains suspended in liquid performed an irregular motion. The motion was later explained by the random collisions with the molecules of the liquid. The motion was describe mathematically by Norbert Wiener who used the concept of a stochastic process $W_t(\omega)$, interpreted as the position at time t of the pollen grain ω. Thus this process is also called Wiener process. See Figure 1 (Techmania - Edutorium, 2008).

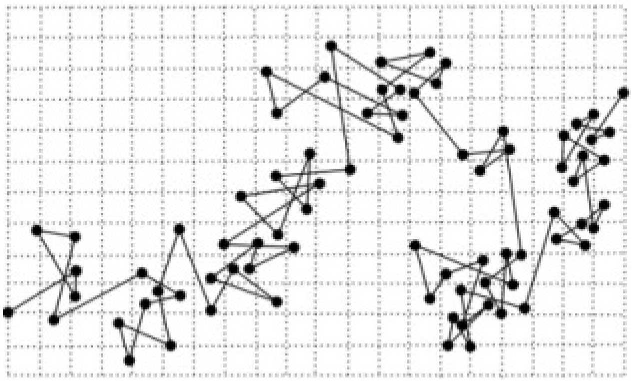

Figure 1: The Brownian motion [12].

Definition 2. The stochastic process W_t is called Brownian motion or Wiener process if the process has some basic properties:

(i) $W_0 = 0$

(ii) $W_t - W_s$ has the distribution $N(0, t-s)$ for $t \geq s \geq 0$

(iii) W_t has independent increments, i.e.

$$W_{t_1},\ W_{t_2} - W_{t_1},\ \ldots,\ W_{t_k} - W_{t_{k-1}}$$

are independent for all $0 \leq t_1 < t_2 \cdots < t_k$.

Note. The unconditional probability density function at a fixed time t

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

The expectation is zero

$$E\left[W_t\right] = 0$$

for $t > 0$.

The variance is t

$$E\left[W_t^2\right] = t.$$

Theorem 1. *Let W_t be Wiener process. Then*

$$E\left[W_t W_s\right] = min\left\{t, s\right\}\ for\, t \geq 0,\ s \geq 0.$$

Proof: [5], pp. 14.

Definition 3. Let $W_i(t), t = 1, 2, \ldots, m$, be a stochastic process. Then $W(t) = (W_1(t), ..., W_m(t))$ denote m-dimensional Wiener process.

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Definition 4. Let $W_t = (W_1(t), ..., W_m(t))$ be m-dimensional Wiener process and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions. Then the process $X_t = (X_1(t), ..., X_m(t), t \in [0,T]$ is the solution of the stochastic differential equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(t,X_t) + \sigma(t,X_t)W_t, \tag{1}$$

$b(t,X_t) \in \mathbb{R}$, $\sigma(t,X_t)W_t \in \mathbb{R}$, where W_t is 1-dimensional "white noise".

After the integration of equation (1) we give the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s,X_s)\mathrm{d}s + \int_0^t \sigma(s,X_s)\mathrm{d}B_s. \tag{2}$$

Equation (2) we can rewrite in the differential form

$$dX_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t. \tag{3}$$

We formally replace the white noise W_t by $\frac{dB_t}{dt}$ and multiply by dt.

4 AN EXISTENCE AND UNIQUENESS OF SOLUTION

Theorem 2. *Let $T > 0$ and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying next conditions:*

(i) Exist some constant C such that

$$\mid b(t,x) \mid + \mid \sigma(t,x) \mid \leq C(1+ \mid x \mid)$$

for $x \in \mathbb{R}^n, t \in [0,T]$.

(ii) Exist some constant D such that

$$\mid b(t,x) - b(t,y) \mid + \mid \sigma(t,x) - \sigma(t,y) \mid \leq D \mid x - y \mid)$$

for $x,y \in \mathbb{R}^n, t \in [0,T]$.

(iii) Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{\mathcal{M}}$ and

$$E\left[\mid Z^2 \mid < \infty\right]$$

Then the stochastic differential equation (3) *has a unique t-continuous solution X_t that*

$$E\left[\int_0^T \mid X_t \mid^2 \mathrm{d}t < \infty\right].$$

Proof: [8], pp. 65.

5 APPLICATIONS

Biology
In biological systems, introducing stochastic noise has been found to help improve the signal strength of the internal feedback loops for balance and other vestibular communication. It has been found to help diabetic and stroke patients with balance control. Many biochemical events also lend themselves to stochastic analysis. Gene expression, for example, has a stochastic component through the molecular collisions — as during binding and unbinding of RNA polymerase to a gene promoter — via the solution's Brownian motion.

Medicine
Stochastic effect is one classification of radiation effects that refers to the random, statistical nature of the damage. In contrast to the deterministic effect, severity is independent of dose. Only the probability of an effect increases with dose.

Epidemic, Population Genetics Process
Markov processes are used to model of epidemic diseases in small populations, among many other phenomena. There exist algorithms like the SSA that simulates a single trajectory with the exact distribution of the process, but it can be time consuming when many reactions take place during a short time interval.

Mathematical Finance
We can suppose a person has an asset or resource (e.g. a house, stocks, oil...) that she is planning to sell. At what time should she decide to sell? Or we can suppose that the person is offered to buy one unit of the risky asset at a specified price. How much should the person be willing to pay for such an option? These and other problems were solved using stochastic analysis.

6 CONCLUSION

Stochasticity is unavoidable when considering biological systems and processes, both at the macro scale with populations surviving in rapidly and unpredictably changing environments, but also and especially at the molecular level, where entropic considerations can have significant implications. Not only must systems be robust but some systems actually rely upon Brownian motions in order to operate efficiently.

The future work will address topics mentioned above.

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