CONTROLLABILITY AND CONTROL CONSTRUCTION FOR LINEAR MATRIX SYSTEMS WITH DELAY

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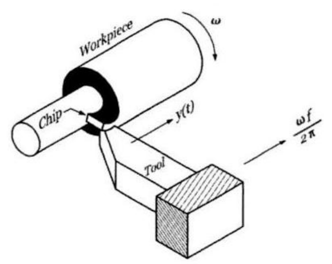
Abstract: Nany researches are devoted for solving problems related with differetial equations with delay, such as, [1]-[5].
In this paper existence of solutions of differential linear matrix equations with delay was investigated. The solutions were found in general form. Necessary and sufficient condition for controllability of differential linear matrix equation with delay was defined and control was built. Paper contains calculated examples.

Keywords: matrix equation with delay, matrix exponential

1 INTRODUCTION

Picture 1. Model of regenerative chatter



Picture 2. Geometry of turning

Model of regenerative chatter [6]
A cylindrical workpiece rotates with constant angular velocity ω and the cutting tool translates along the axis of the workpiece with constant linear velocity $\omega f/2\pi$, where f is the feed rate in length per revolution corresponding to the normal thickness of the chip removed. The tool generates a surface as the material is removed, shown as shaded, and any vibration of the tool is reflected on this surface. In regenerative chatter, the surface generated by the previous pass becomes the upper surface of the chip on the subsequent pass. This time-delay system can be described by the equation

$$m\frac{d^2y(t)}{dt^2}+c\frac{dy(t)}{dt}+ky(t)=-F_t(f+y(t)-y(t-\tau))),$$

where m, c, and k reflect the inertia, damping, and stiffness characteristics of the machine tool, the delay time $\tau=2\pi/\omega$ corresponds to the time for the workpiece to complete one revolution, and $F_t(\cdot)$ is the thrust force depending on the instantaneous chip thickness $f+y(t)-y(t-\tau)$. It is often sufficient to consider $F_t(\cdot)$ to be linear, and techniques for linear time-delay systems are often used

$$F_t(\cdot)=a_1f+a_2y(t)+a_3y(t-\tau),$$

where a_1, a_2, a_3 are constant coefficients. In this case time-delay system can be describes by the equation

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + (k+a_2)y(t) + a_3y(t-\tau) = -a_1 f,$$

or by the matrix linear differential equation with delay

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -\frac{c}{m} & -\frac{k+a_2}{m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{a_3}{m} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix} + \begin{pmatrix} -\frac{a_1}{m} f \\ 0 \end{pmatrix},$$

where $x(t) = \dot{y}(t)$.

2 SYSTEMS WITH DELAY OF THE GENERAL FORM

Let we have the linear matrix differential equation

$$\dot{X}(t) = A_0X(t) + A_1X(t-\tau), \tag{1}$$

with initial condition

$$X(t) = I, \qquad -\tau \leq t \leq 0, \tag{2}$$

where A_0, A_1 are square matrices, I is identity matrix, $\tau > 0, \tau \in R$ is a constant delay.

Theorem 2.1 [8] *Let exists A_0^{-1}. Then the solution of problem (1)- (2) has the recurrence form:*

$$X_{n+1}(t) = e^{A_0(t-n\tau)}X_n(n\tau) + \int_{n\tau}^{t} e^{A_0(t-s)}A_1X_n(s-\tau)ds, \tag{3}$$

where $X_n(t)$ is defined on the interval $(n-1)\tau \leq t \leq n\tau$ and e^{At} is so-called matrix exponential, defined as $e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots$.

Theorem 2.2 [7] *The solution of the equation (1) with not-identity initial conditions $X(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, $X(t) \equiv \Theta$, $t < -\tau$, $\varphi(t)$ is continuous, can be presented in integral form*

$$\mathrm{X_0(t)} = X(\mathrm{t}-\tau)\varphi(0) + \int_{-\tau}^{0} [\mathrm{A_0}X(\mathrm{t}-\tau-\mathrm{s}) + \mathrm{A_1}X(\mathrm{t}-2\tau-\mathrm{s})]\varphi(\mathrm{s})\mathrm{ds}.$$

2.1 NON-HOMOGENEOUS SYSTEM

Let us consider the linear heterogeneous matrix differential equation with delay

$$\dot{X}(t) = A_0X(t) + A_1X(t-\tau) + F(t). \tag{4}$$

Theorem 2.3 [1] *Let A_0 is regular. Then the solution $\overline{X(t)}$ of the heterogeneous equation (4) with zero initial condition, has the form*

$$\overline{X(t)} = \int_0^t \mathrm{X_0}(t-\tau-s)F(s)ds, \quad t \geq 0,$$

where $\mathrm{X_0}(t)$ is the solution of the equation (1) with identity initial condition (2).

As follows from the theory of linear equations, the solution of the non-homogeneous system, satisfying the initial conditions $X(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, is the sum of the homogeneous system, which satisfies these conditions, and the solution the non-homogeneous system that meets the zero conditions.

Theorem 2.4 [1] *The solution of heterogeneous equation (4) with the initial condition $X(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$ has the form*

$$X(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_{0}^{t} \mathrm{X}_0(t-\tau-s)F(s)ds,$$

where $\mathrm{X}_0(t)$ is the solution of the equation (1) with identity initial condition.

3 CONTROLLABILITY RESEARCH

Let X is the state space of dynamic system; U is the set of the controlled effects (controls). Let $x = x(x_0, u, t)$ is the vector that characterizes state of the dynamic system in moment of time t, by the initial condition x_0, $x_0 \in X$, $(x_0 = x|_{t=t_0})$ and by the control function u, $u \in U$.

Definition 3.1 *The state x_0 is called controllable state in the class U (controlled state), if there are exist such control $u, u = u_{x_0} \in U$ and the number T, $t_0 \leq T = T_{x_0} < \infty$ that $x(x_0, u, T) = 0$.*

Definition 3.2 *If every state $x_0, x_0 \in X$ of the dynamic system is controllable, then we say that the system is controllable (controlled system). Similarly relatively controllable system is the dynamic system which every state x_0 is relatively controllable.*

Consider the following Cauchy's problem:

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t),\ t \in [0, T],\ T < \infty, \\ &x(0) = x_0,\ x(t) = \varphi(t),\ -\tau \leq t < 0, \end{aligned} \tag{5}$$

where $x = \{x_1, ..., x_n\}$ is the vector of phase coordinates, $x \in X$, $u(t) = \{u_1(t), ..., u_r(t)\}$ is the control function, $u \in U$, U is the set of piecewise-continuous functions; A_0, A_1, B are constant matrices of dimensions $(n \times n)$, $(n \times n)$, $(n \times r)$ respectively, τ is the constant delay.

Conjecture 3.3 *Because of the view (3) of fundamental matrix of solutions of equation (1) and way of construction of the solution of the heterogeneous equation (4) with the initial condition $X(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, solution of the vector-problem (5) on the time interval $(k-1)\tau \leq x \leq k\tau$ can be written in form*

$$x(t) = f_0(t) + \sum_{p_1=0}^{\infty} A_0^{p_1} B f_1(t,u) + \sum_{p_1=0}^{\infty} A_0^{p_1} A_1 \sum_{p_2=0}^{\infty} A_0^{p_2} B f_2(t,u) + ...$$

$$+ \sum_{p_1=0}^{\infty} A_0^{p_1} A_1 \sum_{p_2=0}^{\infty} A_0^{p_2} A_1 ... A_1 \sum_{p_k=0}^{\infty} A_0^{p_k} B f_k(t,u),$$

where

$$f_{k+1}(t,u) = \int_{k\tau}^{t} \frac{(t-s_1)^{p_1}}{p_1!} \int_{(k-1)\tau}^{s_1-(k-1)\tau} \frac{(s_1-\tau-s_2)^{p_2}}{p_2!} ... \int_{0}^{s_{k-1}-\tau} \frac{(s_{k-1}-\tau-s_k)^{p_k}}{p_k!} u(s_k) ds_k ... ds_1, \quad k > 0,$$

$$f_0(t)=F_{k+1}(t)=e^{A_0(t-k\tau)}\sum_{i=1}^{k}F_i(k\tau)+\int\limits_{k\tau}^{t}e^{A_0(t-s)}A_1\sum_{i=1}^{k}F_i(s-\tau)ds,\quad k\tau<t\le(k+1)\tau,k>0,$$

$$F_1(t)=e^{A_0t}X(0)+\int\limits_{0}^{t}e^{A_0(t-s)}A_1\varphi(s-\tau).$$

Let us consider the control system of differential matrix equation (5).
Now we introduce for the equation (5) analogue of the characteristic equation

$$Q_k(s)=A_0Q_{k-1}(s)+A_1Q_{k-1}(s-\tau),s>0,k=1,2,..\quad Q_0(0)=B,Q_0(s)=0,s\neq0.$$

The first function values are following:

	$s=0$	$s=\tau$	$s=2\tau$	$s=3\tau$	...
$Q_0(s)$	B	0	0	0	0
$Q_1(s)$	A_0B	A_1B	0	0	0
$Q_2(s)$	A_0^2B	$(A_0A_1+A_1A_0)B$	A_1^2B	0	0
$Q_3(s)$	A_0^3B	$(A_0^2A_1+A_0A_1A_0+A_1A_0^2)B$	$(A_0A_1^2+A_1A_0A_1+A_1^2A_0)B$	A_1^3B	0
...	...	...	...	...	...

Let us denote $K=\{Q_0(0),Q_1(0),Q_1(\tau),Q_2(0),Q_2(\tau),Q_2(2\tau),Q_3(0),Q_3(\tau),Q_3(2\tau),Q_3(3\tau),Q_4(0)...\}$, or

$$K=\{B,A_0B,A_1B,A_0^2B,(A_0A_1+A_1A_0)B,A_1^2B,A_0^3B,(A_0^2A_1+A_0A_1A_0+A_1A_0^2)B,$$
$$(A_0A_1^2+A_1A_0A_1+A_1^2A_0)B,A_1^3B,A_0^4B...\}$$

Theorem 3.4 *For controllability of linear stationary system with delay (5) is necessary and sufficient to next condition hold: for $t\ge(k-1)\tau$ is the rank$(K)=n$.*

Proof: The system (5) is relatively controllable when there is exist a control vector $u_0(t)$, that system for some time pass from the initial state $x_0=x(t_0)=(x_1^0,...,x_n^0)^T$ into the direct position $x_1=x(t_1)=(x_1^1,...,x_n^1)^T$. That means, when there is a vector $u_0(t)$, which is performed following equality

$$x(t_1)-x(t_0)=f_0(t_1)+\sum_{p_1=0}^{\infty}A_0^{p_1}Bf_1(t_1,u_0(t_1))+\sum_{p_1=0}^{\infty}A_0^{p_1}A_1\sum_{p_2=0}^{\infty}A_0^{p_2}Bf_2(t_1,u_0(t_1))+...$$
$$+\sum_{p_1=0}^{\infty}A_0^{p_1}A_1\sum_{p_2=0}^{\infty}A_0^{p_2}A_1...A_1\sum_{p_k=0}^{\infty}A_0^{p_k}Bf_k(t_1,u_0(t_1)).\qquad(6)$$

Since $f_0(t_1)$ is a constant vector, that depends on initial conditions and is independent of control (by structure), we introduce such variable $\hat{x}=(\hat{x}_1,..,\hat{x}_n)^T=x(t_1)-x(t_0)-f_0(t_1)$. In the new notation system (6) will be as follows:

$$\hat{x}=\sum_{p_1=0}^{\infty}A_0^{p_1}Bf_1(t_1,u_0(t_1))+\sum_{p_1=0}^{\infty}A_0^{p_1}A_1\sum_{p_2=0}^{\infty}A_0^{p_2}Bf_2(t_1,u_0(t_1))+...$$
$$+\sum_{p_1=0}^{\infty}A_0^{p_1}A_1\sum_{p_2=0}^{\infty}A_0^{p_2}A_1...A_1\sum_{p_k=0}^{\infty}A_0^{p_k}Bf_k(t_1,u_0(t_1)).\qquad(7)$$

In the system (7) we open the sums and regroup terms as follows:

$$\hat{x} = Bg_1 + A_0Bg_2 + A_1Bg_3 + A_0^2Bg_4 + (A_0A_1 + A_1A_0)Bg_5 + A_1^2Bg_6 + A_0^3Bg_7 +$$
$$(A_0^2A_1 + A_0A_1A_0 + A_1A_0^2)Bg_8 + (A_0A_1^2 + A_1A_0A_1 + A_1A_0^2)Bg_9 + A_1^3g_{10} + ... =$$
$$Q_0(0)g_1 + Q_1(0)g_2 + Q_1(\tau)g_3 + Q_2(0)g_4 + Q_2(\tau)g_5 + Q_2(2\tau)g_6 + ...,$$

where $g_1 = f_1(t_1,u_0), p_1 = 0, \quad g_2 = f_1(t_1,u_0), p_1 = 1, \quad g_3 = f_2(t_1,u_0), p_1 = 0, p_2 = 0$
$g_4 = f_2(t_1,u_0), p_1 = 1, p_2 = 0, \quad g_5 = f_2(t_1,u_0), p_1 = 1, p_2 = 1, \quad g_6 = f_3(t_1,u_0), p_1 = 0, p_2 = 0, p_3 = 0....$

Got a system with an infinite number of unknowns and the vector of absolute terms in length n. The system will be the only solution if and only if the rank of the matrix

$$K = \{Q_0(0), Q_1(0), Q_1(\tau), Q_2(0), Q_2(\tau), Q_2(2\tau), ...\}$$

will be equal n, means when the system (7) is linear nonsingular system with n unknowns. In this case the solution of the system will be the vector $(g_1, g_2, ..., g_n)^T$, that is uniquely determined by the vector of absolute terms $\hat{x}$. Since the vector of absolute terms is defined from any finite state of the system (5), we conclude that system (5) can be moved in any point if the conditions of the theorem is true. Its mean that the system (5) is relatively controllable if and only if the matrix K has rank n.

4 FUTURE RESEARCH

In future we are going to build a control which will be optimal due to some criteria.

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