CLOSURE PROPERTIES OF THE FORMAL CONTEXTS

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Abstract: In this paper we study a formal context from the point of view of a topology. In a formal concept analysis basic operator is derivation operator. And here we describe topological properties and objects in terms of derivation operator.

Keywords: Closure, Formal Context, Topology, Saturated set

1 INTRODUCTION

The main idea of the article is an interaction of two branches of mathematics – formal concept analysis and general topology. Formal concept analysis (FCA) was proposed by Rudolf Wille in 1984. Its core is built on applied lattice and order theory. Practical applications were found in different fields including data mining, text mining, machine learning, hierarchical organization of web search results, software development and etc. FCA works with data. And data is described with a binary relationship between an object set and an attribute set. On the other hand, general topology studies properties of topological spaces. The simplest interpretation for an arbitrary topological space we can construct in the next way. An arbitrary topological space (X,τ) we can interpret as a formal context $(X,\tau,\in)$ with the object set X, the attribute set τ and incidence relation $\in$. Many other, more advanced interpretations for a topological space can be found in the literature. On the contrary, from an arbitrary context we can construct in natural way several different topologies. And that topologies deserves a special research.

2 BASIC DEFINITIONS

Definition 1 *(formal context) A formal context is a triple* $(X,A,\vdash)$ *where X, A are sets and* $\vdash \subseteq X\times A$ *is a binary relation between them.*

In formal concept analysis, the elements of X are called *objects* and the elements of A are called *attributes* of the context $(X,A,\vdash)$. The binary relation $\vdash$ is called the incidence relation. We say x has (the attribute) a or x satisfies a.

Definition 2 *(formal concept, extent, intent) Let* $(X,A,\vdash)$ *be a formal context,* $P\subseteq X$*,* $F\subseteq A$*. We put* $P'=\{a|\, a\in A, x\vdash a \text{ for every } x\in P\}$ *and* $F'=\{x|\, x\in X, x\vdash a \text{ for every } a\in F\}$ *. Note: If* $P=\{p\}$ *is a singleton, we simply write* $p'=P'$*. Similarly we write* $f'=F'$ *for* $F=\{f\}$*. The pair* (P,F) *is called a formal concept of the context* $(X,A,\vdash)$ *if* $P'=F$ *and* $F'=P$*. The mappings* $':2^X\to 2^A$ *and* $':2^A\to 2^X$ *are called the derivation operators. We also call P the extent and F the intent of the concept* (P,F)*.*

Now we will define the second derivation operator for a context $(X,A,\vdash)$ (by a composition of the first derivation operators):

(1) Map $'':2^X\to 2^X$ that for $P\in X$, $P\mapsto P''$,

(2) Map $'' : 2^A \to 2^A$ that for $F \in A$, $F \mapsto F''$.

Proposition 1 *(Basic properties) Let $(X,A,\vdash)$ be a context and $M, M_1, M_2 \subseteq X$, $N, N_1, N_2 \subseteq A$ then*

(1) $M_1 \subseteq M_2 \Rightarrow M_2' \subseteq M_1'$, *(1')* $N_1 \subseteq N_2 \Rightarrow N_2' \subseteq N_1'$,

(2) $M \subseteq M''$, *(2')* $N \subseteq N''$,

(3) $M' = M'''$, *(3')* $N' = N'''$.

Definition 3 *(Closure axioms) A closure operator ϕ on a set G is a map assigning a closure $\phi X \subseteq G$ to each subset $X \subseteq G$ if*

(1) $X \subseteq Y \Rightarrow \phi X \subseteq \phi Y$ *for each subset* $X, Y \subseteq G$,

(2) $X \subseteq \phi X$ *for each subset* $X \subseteq G$,

(3) $\phi\phi X = \phi X$ *for each subset* $X \subseteq G$.

An operator ϕ on a set G is a topological closure operator if

(4) $\phi(G \cup M) = \phi G \cup \phi M$.

Second derivation operator is a closure operator. But unfortunately it isn't necessary be a topological closure operator.

Let X be a set, $\zeta \subseteq 2^X$. Let ζ^F be the family of all finite unions of elements of ζ (including the empty union, whose result is $\emptyset$). Then ζ^F is a base for the closed sets of some topology τ on X and ζ is closed subbase; or, in other words, the family $\sigma = \left\{X \smallsetminus P | P \in \zeta^F\right\}$ is an open base for the topology τ.

3 MAIN RESULTS

Definition 4 *(left and right topologies) Let $(X,A,\vdash)$ be a formal context. The topology on X, generated by its closed subbase $\{a' | a \in A\}$ is called the left topology on $(X,A,\vdash)$. Similarly, the right topology on $(X,A,\vdash)$ is the topology on A generated by the family $\{x' | x \in X\}$ used as its subbase for the closed sets.*

Results for the left and right topologies are symmetric. So we will study here only left topology. The topological closure operator defined by this topology we will denote by *cl*.

Definition 5 *A preorder of specialization on a topological space (X,τ) is the binary relation $\leq$ satisfying the condition $x \leq y \Leftrightarrow x \in \mathrm{cl}\{y\}$. We can rewrite this formula as $\mathrm{cl}\{y\} = \downarrow_{\leq} \{y\}$.*

The following question arises in natural way. What is the difference between operators $''$ and *cl*? Before we start to investigate that question we need next theorem.

Theorem 3.1 *Let $(X,A,\vdash)$ be a formal context and τ is its left topology on X, $\mathcal{C}$ is a set of all closed sets in the topological space (X,τ). The following sets are subbases for $\mathcal{C}$ in the topology τ:*

(1) $\mathcal{C}_1 = \{a' | a \in A\}$

(2) $\mathcal{C}_2 = \{F' | F \subseteq A\}$

(3) $\mathcal{C}_3 = \{P | (P,F)$ *is a formal concept of the context* $(X,A,\vdash)\}$

Proof. Let's denote a topology generated by closed subbases $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ correspondingly τ_1, τ_2, τ_3. It is obvious that $\tau = \tau_1$. Let's prove the inclusion $\tau_1 \subseteq \tau_3 \subseteq \tau_2 \subseteq \tau_1$ in succession. If (P,F) is a formal concept of the context $(X,A,\vdash)$ then $P = F' \subseteq X$, $F = P' \subseteq A$. It immediately follows that $\mathcal{C}_3 \subseteq \mathcal{C}_2$ and $\tau_3 \subseteq \tau_2$. Let's take an arbitrary element $a \in A$ and denote $a' = P$, $F = P'$. Than $F' = P'' = a''' = a' = P$. And (a', a'') is a formal concept, so we obtain $\mathcal{C}_1 \subseteq \mathcal{C}_3$ then it follows $\tau_1 \subseteq \tau_3$. It is remained to prove $\tau_2 \subseteq \tau_1$. Let's take $F \subseteq A$ than $F' = \bigcap_{a \in F} a'$. It means that F' is an intersection of sets from $\mathcal{C}_1$ closed in the topology τ_1 and then it is closed in the topology τ. And we proved that $\tau_2 \subseteq \tau_1$. □

Lemma 3.2 *Let* (X,τ) *be a topological space,* $\mathcal{C}$ *be a set of all closed sets,* $\mathcal{C}_2$ *be a subbase for* $\mathcal{C}$ *and* $\mathcal{C}_1$ *be a set of all finite unions of elements* $\mathcal{C}_2$ *(so* $\mathcal{C}_1$ *is a base for* $\mathcal{C}$ *generated from subbase* $\mathcal{C}_2$*). Then for an arbitrary element* $p \in X$ *holds*

$$\mathrm{cl}\{p\} = \bigcap\{C | C \in \mathcal{C}, p \in C\} = \bigcap\{C | C \in \mathcal{C}_1, p \in C\} = \bigcap\{C | C \in \mathcal{C}_2, p \in C\}.$$

Lemma 3.3 *Let* $(X,A,\vdash)$ *be a formal context,* $x, y \in X$*. Then* $x \in y''$ *if and only if* $y' \subseteq x'$*.*

Proof. Let's suppose that $x \in y''$. Then for every $a \in y'$ it holds $x \vdash a$ and it means that $a \in x'$. Then $y' \subseteq x'$. On the other side, let's suppose $y' \subseteq x'$. Let's take an arbitrary element $a \in y'$ then $a \in x'$. It means $x \vdash a$ and then $x \in y''$. □

Theorem 3.4 *Let* $(X,A,\vdash)$ *be a formal context,* τ *be its left topology on* X*. Then for an arbitrary element* $p \in X$ *it holds* $\mathrm{cl}_\tau\{p\} = p''$*.*

Proof. According to Lemma 3.2 it holds $\mathrm{cl}_\tau\{p\} = \bigcap\{a' | p \in a'\}$. Now we need to check relation between p'' and $\bigcap\{a' | p \in a'\}$.

Suppose $x \in p''$. Let's take $a \in A$ that $p \in a'$. Now we need to prove that $x \in a'$. According to Lemma 3.3 we have $p' \subseteq x'$. Besides from formula $p \in a'$ it follows that $a \in p'$, and we can conclude $a \in x'$. It is same as $x \in a'$, then $x \in \bigcap\{a' | p \in a'\}$. It follows that $p'' \subseteq \mathrm{cl}_\tau\{p\}$.

On the other side, suppose $x \in \bigcap\{a' | p \in a'\}$. According to Lemma 3.3 we need to prove that $p' \subseteq x'$. Lets take an element $a \in p'$. It means that $p \in a'$. We know that the set a' is a closed set as an element of closed subbase of topological space (X,τ), it follows $\mathrm{cl}_\tau\{a\} \subseteq a'$. Then $x \in a'$ and it is equivalent to $a \in x'$. Now we can conclude that $p' \subseteq x'$. We checked that $\mathrm{cl}_\tau\{p\} \subseteq p''$. □

Form the previous theorem it follows that on one-element sets topological closure coincide with a second derivation operator.

Corollary 1 *Let* $(X,A,\vdash)$ *be a formal context,* τ *be its left topology on* X*,* $\leq$ *is a preorder of specialization on* X *equipped with topology* τ*. The following statements for arbitrary elements* $x, y \in X$ *are equivalent:*

(1) $x \leq y$,

(2) $x \in \mathrm{cl}_\tau\{y\}$,

(3) $x \in y''$,

(4) $y' \subseteq x'$,

(5) $x'' \subseteq y''$,

The Theorem 3.4 make it possible to construct closure of one-element sets in easy way. But what would happen if we take arbitrary set? The operators $''$ and cl need not necessarily be equivalent for all other sets (See example 1).

Example 1 *Let's take a set $X = \{1,2,3\}$. As a context let's take $(X,X,\Delta X)$ where $\Delta X = \{(1,1),(2,2),(3,3)\}$ is a diagonal relation. It is obvious that left topology τ on X is a discrete topology and $\mathrm{cl}_\tau\{p\} = \{p\} = \{p\}' = \{p\}''$ for every $p \in X$. But for $p \neq q$ it holds $\{p,q\}' = \emptyset$. For example $\{1,2\}'' = \emptyset' = \{1,2,3\} \neq \{1,2\} = \mathrm{cl}_\tau\{1,2\}$.*

Lemma 3.5 *Let $(X,A,\vdash)$ be a formal context. Then every extent is a closed set in the left topology:*

$$Ext(X,A,\vdash) \subseteq \tau^{cl}.$$

A closed set need not necessarily be an extent.

Proof. 1. Directly follows from Theorem 3.1.

2. Counterexample. Lets take a finite context $(\{1,2,3\},\{a,b,c,d\},\vdash)$, where the relation $\vdash$ is represented by the table:

$\vdash$	a	b	c	d
1	x			
2		x	x	x
3			x	

The set of all closed sets is $\tau^{cl} = \{\emptyset,\{1\},\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$. The set of all extents is $Ext(X,A,\vdash) = \{\{1\},\{2\},\{2,3\},\{1,2,3\}\}$. We see, that set $\{1,2\}$ is closed, but it isn't an extent. □

Lemma 3.6 *Let $(X,A,\vdash)$ be a formal context and τ is its left topology, then*

(1) An arbitrary intersection of extents is a closed set.

(2) A finite union of extents is a closed set.

Proof. 1. An arbitrary intersection of extents is an extent and then is a closed set too.

2. A finite union of closed sets is a closed set. □

Theorem 3.7 *Let $(X,A,\vdash)$ be a formal context and τ is its left topology. Let's denote $Ext^F(X,A,\vdash)$ the set of all finite unions of extents. Then:*

$$Ext^F(X,A,\vdash) \subseteq \tau^{cl}.$$

If set X is finite, then

$$Ext^F(X,A,\vdash) = \tau^{cl}.$$

Proof. The inclusion for the infinite case is obvious (it directly follows from the Lemma 3.5). Let's have a look at the second part. $Ext^F(X,A,\vdash) \subseteq \tau^{cl}$ is obviously true. Now it remains to prove $\tau^{cl} \subseteq Ext^F(X,A,\vdash)$. Let Y be an arbitrary closed set in the left topology. Then by the Theorem 3.1 of the left topology we can easily obtain $Y = \cap_{J_{arb}} \cup_{I_{fin}} M'_{i,j}$ where $M_{i,j} \in Ext(X,A,\vdash)$. Because X is a finite set, then J_{arb} is a finite set too. So we have $Y = \cap_{J_{fin}} \cup_{I_{fin}} M_{i,j} = \cup_{I_{fin}} \cap_{J_{fin}} M_{i,j}$. It means, that every closed set we can represent as a finite union of some extents. □

Definition 6 *Let* $(X,A,\vdash)$*be a topological space. A set* $B \subseteq X$ *is a saturated set in the topology* τ *on* X *if it is an intersection of open sets. If a binary relation* $\leq$ *is the preorder of specialization on the topological space* (X,τ)*, then set* B *is saturated iff* $B = \uparrow_{\leq} \{B\} = \{x | x \in X, \exists a \in B : a \leq x\}$*.*

Theorem 3.8 *Let* $(X,A,\vdash)$ *be a formal context and* (X,τ) *be a left topology on it. Then for an arbitrary set* $P \subseteq X$ *the following statements are equivalent:*

(1) P *is a saturated set,*

(2) $P = \uparrow_{\leq} \{P\}$*,*

(3) $\{x | x \in X \setminus P, P \cap x'' \neq \emptyset\} = \emptyset$*,*

(4) $\forall x \in X \setminus P, P \cap x'' = \emptyset$*.*

Proof. $(1) \Leftrightarrow (2)$ was mentioned above.

$(2) \Leftrightarrow (3)$ $\uparrow_{\leq} \{P\} = \{x | x \in X, \exists a \in P : a \leq x\} = \{x | x \in X, \exists a \in P : a \in x''\} = \{x | x \in X, P \cap x'' \neq \emptyset\} = \{x | x \in P, P \cap x'' \neq \emptyset\} \cup \{x | x \in X \setminus P, P \cap x'' \neq \emptyset\} = P \cup \{x | x \in X \setminus P, P \cap x'' \neq \emptyset\}$. We can conclude that $\{x | x \in X \setminus P, P \cap x'' \neq \emptyset\} \subset P$ must hold, because $P \cup \{x | x \in X \setminus P, P \cap x'' \neq \emptyset\} = P$. But there is only one possibility $\{x | x \in X \setminus P, P \cap x'' \neq \emptyset\} = \emptyset$ (because obviously it is a subset of $X \setminus P$).

$(3) \Leftrightarrow (4)$ is obvious. □

4 CONCLUSION

In this paper we started a study of an informational structure named formal context in terms of topology. For the finite case, the topologically closed sets are generated as unions of extents. We also described the relationships between closure operator, second derivation operator and saturation.

5 ACKNOWLEDGEMENT

This article was supported by the International Visegrad Fund.

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