LAGUERRE FUNCTIONS IN ELECTRICAL ENGINEERING

Martin Tůma
Doctoral Degree Programme (2), FEEC BUT
E-mail: xtumam02@stud.feec.vutbr.cz

Supervised by: Pavel Jura
E-mail: jura@feec.vutbr.cz

Abstract: This contribution deals with the use of the Laguerre functions in electrical engineering. After the short introduction the definition of the Laguerre polynomials and functions is given. The application of the discrete Laguerre transform on the data compression is shown. It is pointed out that the discrete Laguerre transform can give better results than discrete cosine transform in the task of the data compression.

Keywords: Laguerre, compression, polynomial

1 INTRODUCTION

The Laguerre functions were introduced by Edmond Laguerre more than 150 years ago. Many applications of them on various problems in mathematics, physics and electrotechnics were found. The main aim of this work is to show some possible application in electrical engineering. On the two examples it will be shown how can the usage of Laguerre functions help to get better results in the data compression task.

2 THE LAGUERRE POLYNOMIALS AND FUNCTIONS

In the following some basic definitions will be presented. The generalized Laguerre polynomial $l_n^a(x)$ are the solutions of the following differential equation

$$xy'' + (a+1-x)y' + ny = 0, \qquad n \in \mathbb{N}_0, a \in (-1,\infty).$$

The generalized Laguerre polynomials can be written in the following form

$$l_n^{(a)}(x) = \frac{x^{-a}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+a}).$$

The generalized Laguerre polynomials form the complete orthogonal system in $L_2(0,\infty)$ with respect to the weight function $x^a e^{-x}$, i.e.,

$$\int_0^\infty l_i^{(a)}(x) l_j^{(a)}(x) x^a e^{-x} dx = \binom{n+a}{n}\Gamma(a+1)\delta_{i,j}.$$

The so-called simple Laguerre polynomials can be found in the literature. These can be obtained simply by putting $a = 0$,

$$l_n^{(0)}(x) = l_n(x).$$

In the following figure the first 5 Laguerre polynomials are shown:

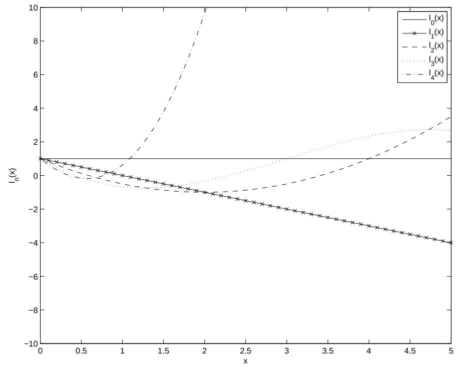


Figure 1: Laguerre polynomials

The orthonormalized Laguerre polynomials are called the Laguerre functions $L_n^{(a)}(x)$,

$$L_n^{(a)}(x) = \sqrt{\frac{n!x^a}{\Gamma(n+a)e^x}}\, l_n^{(a)}(x).$$

The special case for $a = 0$ is $L_n^{(0)}(x) = e^{-\frac{x}{2}} l_n^{(0)}(x)$. These simple Laguerre functions are used in the experiments bellow.

There are many reference books and articles about the theory of Laguerre polynomials and functions. Their main application is in the field of identifying the dynamical systems. In [3] there is the simple example of the black box model identifying based on the transform to the Laguerre functions basis. However, it's possible that their potential for practical computation wasn't fully exploited yet due to the numerical problems, which occurred during their implementation. As pointed out in the fairly new article [4] the source of the numerical problems is mainly in the usage of Laguerre polynomials instead of the Laguerre functions. The Laguerre polynomials are usable only inside small intervals due to their extremely ill-conditioned behavior. The basis of the Laguerre functions for the discrete Laguerre transform will be used in the next section.

3 DISCRETE LAGUERRE AND COSINE TRANSFORMS

In this section the comparison between the discrete Laguerre and cosine transforms (DLT, DCT) will be presented.

The DCT was introduced in 1974 into electrical engineering literature by N. Ahmed, T. Natarajan and K.R. Rao in their article [2]. It is the real version of the discrete Fourier transform. Nowadays it is the core of many algorithms for data compression and signal processing. For example, DCT is used in the JPG and MP3 algorithms for image and sound processing.

Although there are many articles about the Laguerre polynomials and functions, the transform similar to DCT based on the Laguerre orthonormal functions wasn't introduced till 1995 when the article [1] appeared. In this article the DLT was defined with the help of Gauss-Laguerre integration in the similar way as the other orthonormal transforms. It was suggested, that this transform could lead to the better results in the data compression tasks especially for the vectors, that decay exponentially to zero, i.e., that have the same behavior as the Laguerre basis functions.

Now the following data compression task for $z \in \mathbb{R}^N$ will be presented. Let's consider the Fourier expansion for the vector z, i.e.,

$$z = \sum_{i=1}^{N} c_i u_i,$$

where $\{c_i\}$ are the Fourier coefficients for some orthonormal basis $\{u_i\}$ of $\mathbb{R}^N$. Now consider the truncated expansion for some $K \leq N, K \in \mathbb{N}$, i.e.,

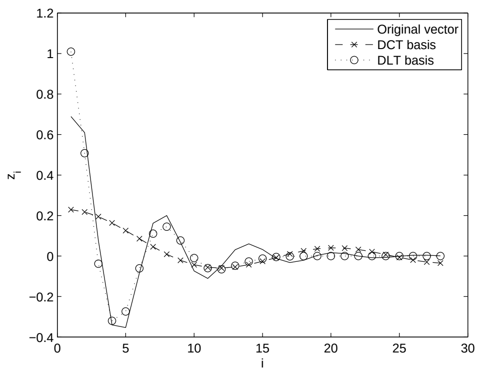
$$w = \sum_{i=1}^{K} c_i u_i.$$

The vector reconstruction w is the approximation of the vector z. This move from the vector z to the vector w is often called the compression of the vector z or simply the reduction of the model. The main idea of this compression is that the most of the information in the vector is contained in the first few Fourier coefficients of the vector expansion.

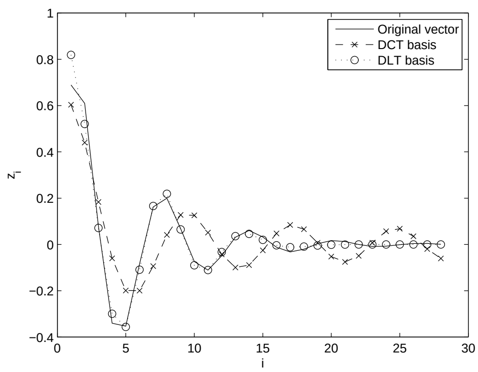
In the following there are the pictures of the vector of length $N = 28$ reconstruction for $K = 4, 8, 24$ and the graphs and tables of the relative compression error $\|z - w\|/\|z\|$ for $K = 4, 8, 12, 16, 20, 24$ shown. All experiments were done in MATLAB.

The first vector is the exponentially damped sinusoid sampled in the interval $[0, 2.8]$, i.e.,

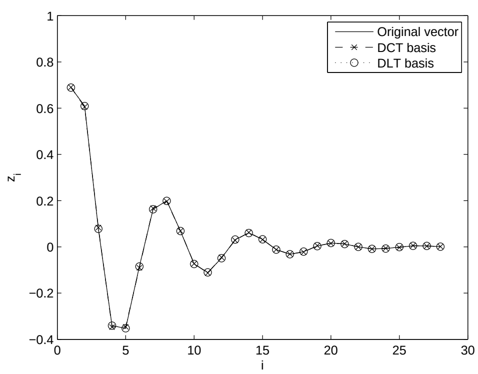
$$z_i = e^{-0.2i} \sin(i), \qquad i = 1..28.$$



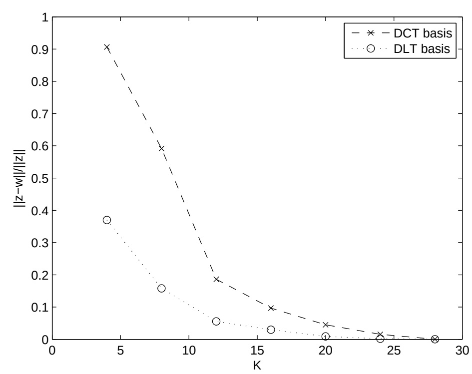
(a) Reconstruction of the first vector for $K = 4$



(b) Reconstruction of the first vector for $K = 8$



(c) Reconstruction of the first vector for $K = 24$



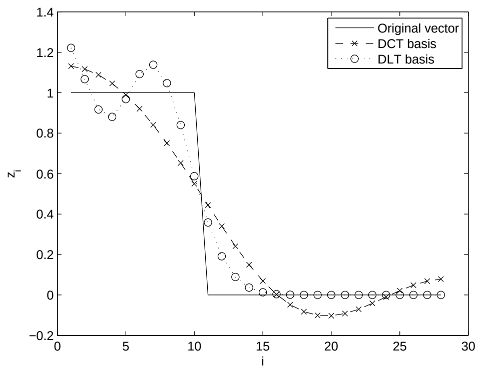
(d) Relative compression error of the first vector

K	4	8	12	16	20	24
DCT basis	0.9066	0.5918	0.1861	0.0968	0.0450	0.0156
DLT basis	0.3700	0.1578	0.0555	0.0296	0.0089	0.0019

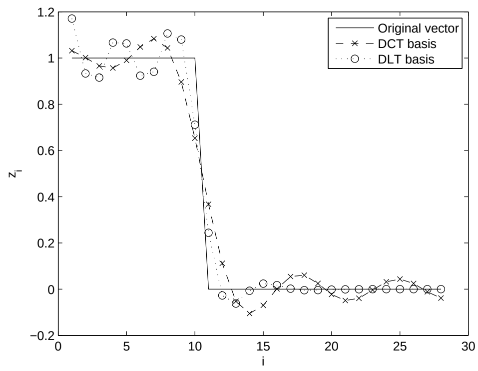
Table 1: Relative compression error of the first vector

The second vector is the unit descent function sampled in the interval $[0, 2.8]$, i.e.,

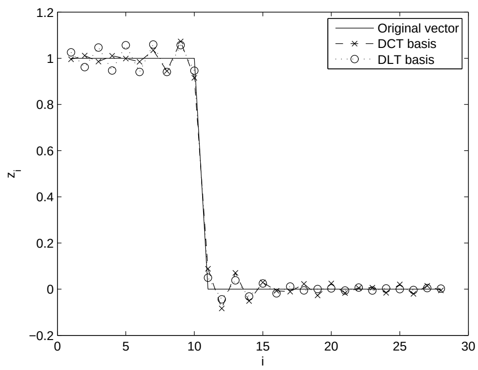
$$z_i = \begin{cases} 1 & 1 \le i \le 10, \\ 0 & 11 \le i \le 28. \end{cases}$$



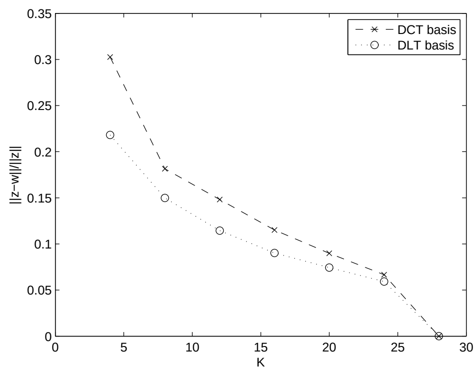
(e) Reconstruction of the second vector for $K = 4$



(f) Reconstruction of the second vector for $K = 8$



(g) Reconstruction of the second vector for $K = 24$



(h) Relative compression error of the second vector

K	4	8	12	16	20	24
DCT basis	0.3026	0.1815	0.1481	0.1152	0.0898	0.0666
DLT basis	0.2181	0.1499	0.1144	0.0902	0.0744	0.0592

Table 2: Relative compression error of the second vector

4 CONCLUSION

In the two examples above it was presented that the DLT performs significantly better than the DCT in the term of the relative compression error. The future work may be focused on the searching for the precise definition of the classes of functions for which the use of DLT can bring such good results and on the computational aspects of DLT. Also the use of the generalized Laguerre functions with different values of the parameter a and their comparison to the simple Laguerre functions could be interesting and it could lead to even better results.

ACKNOWLEDGEMENT

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