HYPERGROUPS OF SECOND-ORDER DIFFERENTIAL OPERATORS IN THE JACOBI FORM AND MULTI-QUASIAUTOMATA

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ABSTRACT

We study multistructures of linear second-order differential operators which can be used for construction of multiautomata serving as a theoretical background for modelling of processes.

1 INTRODUCTION

The contribution is devoted to an algebraic approach to linear homogeneous differential equations, in particular to systems of differential operators forming their left hand sides. The presented analysis is based on two aspects. Firstly, on ideas of the classical school of Professor Otakar Borůvka developing global theory of linear differential equations and their transformations widely applying algebraic, geometrical and topological approaches to the mentioned topics – [11, 12] (see also many other papers of Professor František Neuman). Algebraic approach of Borůvka's school has used classical group theory. Since the beginning of the first decade of this century relationships between ordinary linear differential operators and the hypergroup theory – have been studied - cf. [2, 3, 6, 8] Further, a certain construction of algebraic binary hyperstructures (semihypergroups and hypergroups) from ordered algebraic systems is based on a certain lemma on principal ends generated by products of pairs of elements know as Ends-lemma. First version of this lemma was obtained in the monography [4]; for furter reading cf. [13, 14, 15]. Constructions of this type yield the way to obtain semihypergroups and hypergroups of linear differential operators – ordinary differential operators of a given order n or partial differential operators and mentioned constructions allow also to create hyperstructures of integral operators of Fredholm or Volterra-type. In the present contribution we construct actions of commutative transposition hypergroups i.e. join spaces created from rings of continuous and smooth functions of a given class on semihypergroups or hypergroups of second order linear ordinary differential operators. These constructed structures are in fact discrete dynamical systems with a phase (additive) hypergroups of continuous and smooth functions and phase set formed by the above mentioned differential operators. As a suitable synonyma we can use a multiautomaton without output function, where the transition function or next state function satisfies so called Generalized Mixed Asociativity Condition (GMAC) – see bellow. These structures are also termed as quasiautomata or multi-quasiautomata.

Let J be an open interval of real numbers, $\mathbb{C}(J)$ be the ring of all continuous functions on J and

$\mathbb{C}^+(J)$ its subsemirings of all positive functions. In what follows we denote $L(p,q)y = y'' + p(x)y' + q(x)y$ and $y'' + q(x)y = 0; q \in \mathbb{C}(J)$, i.e. $L(0,q)y = 0$. Otakar Borůvka has obtained a criterion of a global equivalence for second order differential equations within the Jacobi form, i.e.

$$y'' + q(x) \cdot y = 0,\ q \in \mathbb{C}(J)$$

and he also found corresponding global canonical forms for such equations. We will consider the bellow defined operation on the set of such differential operators (under the supposition $q(x) \neq 0,\ x \in J$). Denote by $\mathbb{JA}_2(J)$ a subset of $\mathbb{LA}_2(J)$ defined

$$\mathbb{JA}_2(J) = \{L(0,q);\ q \in \mathbb{C}^+(J),\ q(x) \neq 0, x \in J\}.$$

The following theorem is proved in [1] as Theorem 1.

Theorem 1.1 *Let $J \subseteq \mathbb{R}$ be an open interval,* $\mathbb{LA}_2(J) = \{L(p,q);\ p,q \in \mathbb{C}(J),\ q(x) \neq 0, x \in J\}$, $\mathbb{JA}_2(J) = \{L(0,q);\ q \in \mathbb{C}^+(J),\ q(x) \neq 0, x \in J\}$.

For any pair of operators $L(p_1,q_1),\ L(p_2,q_2) \in \mathbb{LA}_2(J)$ we define

$$L(p_1,q_1) \bullet_B L(p_2,q_2) = L(p_1q_2 + p_2, q_1q_2).$$

Then $(\mathbb{LA}_2(J)_q, \bullet_B)$ is a non-commutative group, $(\mathbb{JA}_2(J)_q, \bullet_B)$ is its commutative subgroup with the unit element $L(0,1)$ which assigns to any function $f \in \mathbb{C}^2(J)$ the function $f'' + f$.

2 USED CONCEPTS

Recall some basic notions and notation of the hypergroup theory from c.f. [4, 5, 7, 8]. *A hypergroupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \longrightarrow \mathcal{P}^*(H)$ is binary hyperoperation on H. (here $\mathcal{P}^*(H)$ denotes the system of all nonempty subset of H). If $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a,b,c \in H$ then $(H, \bullet)$ is called a *semihypergroup.* If moreover the reproduction axiom ($a \bullet H = H = H \bullet a$ for any element $a \in H$,) is satisfied, then the pair $(H, \bullet)$ is called a *hypergroup*.

A hypergroup $(H, \bullet)$ is called *a transposition hypergroup* or *a join space* if it satisfies the transposition axiom: *For $a,b,c,d \in H$ the relation $b \setminus a \approx c/d$ implies $a \bullet d \approx b \bullet c$ (here $X \approx Y$ for $X,Y \subseteq H$ means $X \cap Y \neq 0$), where sets $b \setminus a = \{x \in H;\ a \in b \bullet x\}, c/d = \{x \in H;\ c \in x \bullet d\}$ are called left and right extensions or fractions in the given order.* Clearly, if hyperoperation "$\bullet$" is commutative, the fractions $b \setminus a, a/d$ coincide.

By a multi-quasiautomaton we mean a triad (A, S, δ) where A is semi-hypergroup or a hypergroup, S is a non-empty set and $\delta :\ A \times S \rightarrow S$ is a transition map satisfying these conditions:

1) $\delta(e,s) = S$ for any $s \in S$ and the identity $e \in A$, if it exists (the identity condition)
2) $\delta(b, \delta(a,\ s)) \in \delta(a \cdot b,\ s)$ for all $a,\ b \in A,\ s \in S$ (the generalized mixed associativity condition - GMAC).

3 CONSTRUCTION OF THE BASIC HYPERGROUP OF DIFFERENTIAL OPERATORS

We denoted by $\mathbb{LA}_2(J)$ the set of all such differential operators by considered in Theorem 1.1 (second coefficients q are positive functions). On this set we define a binary operation:

$$L(p_1,q_1) \bullet_B L(p_2,q_2) = L(p_2 + p_1q_2,\ q_1q_2)$$

This operation satisfies the associativity axiom: $L(p_1,q_1)\bullet_B(L(p_2,q_2)\bullet_B L(p_3,q_3)) =$ $= L(p_1,q_1)\bullet_B L(p_3+p_2q_3,q_2q_3) = L(p_3+p_2q_3+p_1q_2q_3,q_1q_2q_3) = L(p_2+p_1q_2,q_1q_2)\bullet_B$ $\bullet_B L(p_3,q_3) = (L(p_1,q_1)\bullet_B L(p_2q_2))\bullet_B L(p_3q_3)$. Since$(L(p_1,q_1)\bullet_B L(0,1) = L(p_1,q_1) =$ $= L(0,1)\bullet_B L(p_1,q_1))$, the operator $L(0,1)$ is a neutral element of $(\mathbb{LA}_2(J),\ \bullet_B)$. Further $L(p_1,q_1)\bullet_B L(\frac{-p_1}{q_1},\frac{1}{q_1}) = L(0,1) = L(\frac{-p_1}{q_1},\frac{1}{q_1})\bullet_B L(p_1,q_1)$, hence $L(\frac{-p_1}{q_1},\frac{1}{q_1})$ is the inverse element to $L(p_1,q_1)$; thus evidently $(\mathbb{LA}_2(J),\bullet_B)$ is a non-commutative group.

Now we define on the set $\mathbb{LA}_2(J)$ a hyperoperation:

$$L(p_1,q_1)\ *\ L(p_2,q_2) = \{L(p_2+p_1q_2,\ \varphi); q_1(x)q(x)\le\varphi(x); \varphi\in\mathbb{C}(J)\}$$

Then:

1) $\mathbb{LA}_2(J)*\mathbb{LA}_2(J) \subseteq \mathbb{LA}_2(J)$, i.e.$(\mathbb{LA}_2(J),\ *)$ is a hypergoupoid.
   Indeed if $L(p_1,q_1),L(p_2,q_2)\in\mathbb{LA}_2(J)$, then $L(p_1,q_1)*L(p_2,q_2) = \{L(u,v);$ $L(p_1,q_1)\cdot L(p_2,q_2)\le L(u,v)\} = \{L(u,v);\ L(p_2+p_1q_2,q_1q_2)\le L(u,v)\} =$ $= \{L(p_2+p_1q_2,v);\ q_1(x)q_2(x)\le v(x),\ x\in J\}\subseteq\mathbb{LA}_2(J)$

2) After some effort we can verify validity of the reproduction axiom. Hence $(\mathbb{LA}_2(J),*)$ is a non-commutative hypergroups.

4 SUBHYPERGROUPS OF DIFFERENTIAL OPERATORS IN JACOBI FORM

The set $\mathbb{JA}_2(J)$ is a set of differential operators in the Jacobi form. On this set we define hyperoperation "$*$" by the rule for: $L(0,p),\ L(0,q)\in\mathbb{JA}_2(J)$, we put $L(0,p)*L(0,q) = \{L(0,\varphi); p(x)\cdot q(x)\le\varphi(x);\ \varphi\in\mathbb{C}(\mathbb{J})\}$. This structure is a commutative semihypergroup.

If we suppose $\varphi>0$, *i.e.* $\varphi\in\mathbb{C}^+(J)$, then the obtained semihypergroup satisfies the reproduction axiom. Hence $(\mathbb{JA}_2(J),\ *)$ is a commutative hypergroup. If we denote $\mathbb{J}_c\mathbb{A}_2(J) = \{L(0,r);$ $r\in\mathbb{R},\ r\neq 0\}$, then with respect to Theorem 2 [1] we have that subhypergroupoid $(\mathbb{J}_c\mathbb{A}_2(J),*)$ of the group $(\mathbb{JA}_2(J),\ *)$ is its normal commutative subgroup. Thus can be easily verify.

Indeed since for every pair of operators $L(0,r),\ L(0,s)\in\mathbb{J}_c\mathbb{A}_2(J)\subset\mathbb{JA}_2(J)$ there holds $L(0,r)*_c$ $*_c L(0,s) = \{L(0,t);\ t\in\mathbb{R}^+; r\cdot s\le t\} = L(0,r)*L(0,s) =$
$\{L(0,\varphi);\ \varphi\in\mathbb{C}_+(J);\ r\cdot s\le\varphi(x),\ x\in J\}\ \cap\mathbb{J}_c\mathbb{A}_2(J)$ (hence $\varphi\in\mathbb{C}_+(J)$ is positive constant function only which can be identify with its value) we have that$(\mathbb{J}_c\mathbb{A}_2(J),\ *_c)$ is a subhypergroupoid of the hypergroup $(\mathbb{JA}_2(J),*)$. The hyperoperation "$*_c$" is evidently associative and commutative. Similarly as above (using positive real numbers representing positive constant function only) we obtain that the inclusion $(\mathbb{J}_c\mathbb{A}_2(J)\subset L(0,r)\ *_c\mathbb{J}_c\mathbb{A}_2(J)$ is valid for arbitrarily chosen operator $L(0,r)\in(\mathbb{J}_c\mathbb{A}_2(J)$. Since the opposite inclusion is evident, we have that the semihypergroup $(\mathbb{J}_c\mathbb{A}_2(J),\ *_c)$ is a commutative subhypergroup of the hypergroup $(\mathbb{JA}_2(J),\ *)$. Notice that the hypergroup $(\mathbb{J}_c\mathbb{A}_2(J),\ *_c)$ is closed and reflexive; for the prof see [5].

5 QUASIAUTOMAT WITH STATES FORMED BY DIFFERENTIAL OPERATORS IN JACOBI FORM

Using binary hyperstructures with one hyperoperation, in particular semihypergroups or hypergroups we can construct multi-quasiautomaton with a given hyperstructure as an input alphabet.

These multi-quasiautomata (see above) are generalizations of the classical concept of automata serving for modelling of various processes.

We will construct these a structures using differential operators is the Jacobi form. Another approach illustrates the following construction. Consider a structure $((\mathbb{C}_+(J)_0,\ \triangle), \mathbb{JA}_2(J),\ \delta_{\mathbb{J}})$, where: $f \triangle g\ =\ (\bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+}[af\ +\ bg)_{\leq}$. By theorem 3.2 [6] the hypergroupoid $(\mathbb{C}_+(J)_0,\ \triangle)$ is a hypergroup satisfying transposition axiom, thus it is a join space [2]. Recall that $\mathbb{C}_+(J)_0\ =$ $=\{f : J \to \mathbb{R};\ f(x) \geq 0,\ x \in J\}$.
Define $\delta_{\mathbb{J}} :\ \mathbb{C}_+(J) \times \mathbb{JA}_2(J) \to \mathbb{JA}_2(J)$ by $\delta_{\mathbb{J}}(f,\ L(0,q))\ =\ L(0,f+q)\ =\ L(0,f)\cdot L(0,q)$

$$1)\quad \delta_{\mathbb{J}}(0,\ L(0,q))\ =\ L(0,q)$$
$$2)\quad \delta_{\mathbb{J}}(g,\ \delta_{\mathbb{J}}(L(0,q))) \in \delta_{\mathbb{J}}(f\ \triangle g,\ L(0,q))$$

Left hand side: $\delta_{\mathbb{J}}(g,\ \delta_{\mathbb{J}}(L(0,q)))\ =\ \delta_{\mathbb{J}}(g,\ L(0,\ f+q))\ =\ L(0,\ f+g+q)$

Right hand side: $\delta_{\mathbb{J}}(f\ \triangle g,\ L(0,q))\ =\ \delta_{\mathbb{J}}(\bigcup_{[a,b]\in\mathbb{R}_0^+\times\mathbb{R}_0^+}[af\ +\ bg)_{\leq},\ L(0,q))\ =\ \{L(0,q\ +\ q);$ $\exists[a,b] \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ :\ af\ +\ bg \leq \varphi\}$ For $a\ =\ 1\ =\ b, \varphi\ =\ f+g$ we have $L(0,q+\varphi) \in$ $\delta_{\mathbb{J}}(f \triangle g,\ L(0,q))$,

i.e. GMAC is satisfied. So, the structure $((\mathbb{C}_+(J)_0,\ \triangle), \mathbb{JA}_2(J),\ \delta_{\mathbb{J}})$ is a multi-quasiautomaton.

It is be noted that using the group of operators $(\mathbb{JA}_2(J),\ \bullet_B)$ we can construct an action of the additive group of all integer on the phase space $\mathbb{JA}_2(J)$ (or in terminology of O.Borůvka - an algebraic space with operators).
Let $(\mathbb{Z},\ +)$ be the a additive group of all integers. Let $L(p,q) \in \mathbb{JA}_2(J)$ be arbitrary but fixed chosen differential operator. Denote by $\bigwedge_q\ : \mathbb{JA}_2(J) \longrightarrow \mathbb{JA}_2(J)$ the left translation determined by $L(p,q)$, i.e. $\bigwedge_q(L(p,q)) = L(p,q)\ \bullet_B\ L(p,q))$ for any operator $L(p,q) \in (\mathbb{JA}_2(J)$. Further, denote by $\bigwedge_q^r$ the r-th iteration of $\bigwedge_q$ for $r \in \mathbb{Z}$. Now define $\pi_q\ : \mathbb{JA}_2(J) \times \mathbb{Z} \longrightarrow \mathbb{JA}_2(J)$ by $\pi_q(L(p,q),r) = \bigwedge_q^r(L(p,q))$ It is easy to see that we get a usual (discrete) transformation group, i.e. the action of $(\mathbb{Z},\ +)$ (as the phase group) onto $(\mathbb{JA}_2(J))$, thus the following two requirements are satisfied:

1. $\pi_q(L(p,q),0)\ =\ L(p,q)$ for any $L(p,q) \in \mathbb{JA}_2(J)$,
2. $\pi_q(L(p,q),r\ +\ s)\ =\ \pi_q(\pi_q(L(p,q),r),s)$

for any operator $L(p,q) \in \mathbb{JA}_2(J)$ and any pair of integers $r,s \in (\mathbb{Z},\ +)$

6 CONCLUSIONS

The presented theory enlarging algebraic approach to differential equations due certain part of Borůvka's school serves as the mathematical backround for the analysis of various processes, e.g. of systems containing continuous time functions representing continuous time signals or non-periodic impulses - represented by general polynomials. Some application possibilities are open.

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