A PROGRESS IN EXISTENCE OF UNDOMINATED STRATEGIES

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Abstract: In this paper we report a certain progress we reached in a problem of existence of undominated strategies in normal form games. We have improved and generalized a well-known theorem of Herve Moulin ensuring the existence of undominated strategies in a normal form game if the set of strategies of a player is compact and his utility function is continuous.

Keywords: Normal form game, undominated strategy, net, (almost) compact topological space

1 INTRODUCTION

Undominated strategies play an important role in game theory as well as in many related engineering and economical applications. The theorem ensuring the existence of undominated strategies in a normal form game under the assumption that the set of all strategies of a player is compact and the utility function is continuous, belongs to the well-known and fundamental results. Perhaps it could be difficult to say when the result was published first – at least, it was stated in 1981 in Herve Moulin's comprehensive textbook on game theory [8], and essentially it was also contained and used in many other papers. The proof presented in the first edition of [8] was dependent on a combination of relatively non-trivial results from measure theory, metric topology and mathematical analysis. In the second, revised edition [9] of the same book, now there is stated a simplified proof using some topological argumentation together with Zorn's Lemma. However, the proof in [9] is unfortunately incorrect, since it implicitly uses a non-valid argument that every chain (that is, a linearly ordered set) contains a cofinal subsequence. The first uncountable ordinal ω_1 is a proper counterexample witnessing that in general it is not true. The mistake itself is not very critical for game theory, since in metric spaces, for which the classical results are usually formulated, the topology is first countable and hence the sequences are still sufficient to fully describe the topology by means of the convergence. Nevertheless, the mentioned fact itself, was a source of inspiration for a revision of of the original Moulin's Theorem leading to its our generalization and improvement. A natural question how substantial our improvement really is we will demonstrate on a simple example.

2 DEFINITIONS AND DENOTATIONS

Recall that an n-person game G in a *normal* or *strategic* form is denoted by the $2n$-tuple $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$, where for each $i \in \{1, 2, \ldots, n\}$, X_i is a non-empty set of strategies of the i-th player and $u_i : \Pi_{j=1}^{n} X_j \to \mathbb{R}$ is his real valued *utility*, or *pay-off* function. Let $i \in \{1, 2, \ldots, n\}$ and let $x_i, y_i \in X_i$ be some strategies of the i-th player. We say that the strategy y_i dominates the strategy x_i, if the following conditions holds:

(1) For any selection of strategies $s_k \in X_k$, where $k \in \{1, 2, \ldots, n\}$, $k \neq i$,

$$u_i(s_1, s_2 \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n) \leq u_i(s_1, s_2 \ldots, s_{i-1}, y_i, s_{i+1}, \ldots, s_n).$$

(2) For each $k \in \{1,2,\ldots,n\}$, $k \neq i$, there exists some strategy $t_k \in X_k$ such that

$$u_i(t_1,t_2\ldots,t_{i-1},x_i,t_{i+1},\ldots,t_n) < u_i(t_1,t_2\ldots,t_{i-1},y_i,t_{i+1},\ldots,t_n).$$

The strategy $x_i \in X_i$ of the i-th player is said to be *undominated* if there is no strategy $y_i \in X_i$ which dominates x_i. It should be noted that this kind of dominance is sometimes referred as a *weak dominance*, in opposite to the *strict dominance*, which differs from the above defined notion at the condition (1) by the strict form $<$ of the inequality. Two strategies $x_i, y_i \in X_i$ are called *equivalent*, if for any selection of strategies $s_k \in X_k$, where $k \in \{1,2,\ldots,n\}$, $k \neq i$, it holds

$$u_i(s_1,s_2\ldots,s_{i-1},x_i,s_{i+1},\ldots,s_n) = u_i(s_1,s_2\ldots,s_{i-1},y_i,s_{i+1},\ldots,s_n).$$

(For more detail, see, for example, [2], [11].)

A binary relation on a set is called a *preorder*, if it is reflexive and transitive (and not necessarily antisymmetric). Let A be a non-empty set, $\preccurlyeq$ be a preorder on A such that for every $x,y \in A$ there exists $z \in A$ with $x \preccurlyeq z$ and $y \preccurlyeq z$. Then we say that $(A,\preccurlyeq)$ is a *directed* set. A *net* in a topological space X is an arbitrary mapping from a directed set to the space X. Recall that if $f : X \to Y$ is a continuous mapping between topological spaces X, Y and φ is a net in X, having a cluster point $x \in X$, then $f \circ \varphi$ is a net in Y, having the corresponding cluster point $y = f(x) \in Y$. A family Φ of non-empty sets is called a filter base if any intersection of two sets belonging to Φ contains a subset from Φ. Let X be a topological space. We say that $p \in X$ is a θ-cluster point of a filter base Φ in X, if for every closed neighborhood H of p and every $F \in \Phi$, the intersection $H \cap F$ is non-empty. Similarly, p is a θ-cluster point of a net $\varphi(A,\preccurlyeq)$, if for each closed neighborhood H of p and for each $a \in A$, there exists $b \in A$, $b \succcurlyeq a$, such that $\varphi(b) \in H$. Taking the φ-images of the principal upper sets $\uparrow a = \{b | \, b \in A, b \succcurlyeq a\}$ one can easily convert the net $\varphi(A,\preccurlyeq)$ into a filter base, while the corresponding convergence and θ-convergence notions will be preserved.

A topological space X is said to be *compact*, if every net or every filter base in X has a cluster point. For more detail and other equivalent and well-known characterizations of compactness, especially in terms of open covers, we refer the reader to [3]. We also remark that in a modern approach to compactness, motivated by the growing interest of the theoretical computer scientists in topology, the Hausdorff separation axiom is no longer assumed as a part of the definition of compactness (see, for example, [16]). Recall that a topological space is *almost compact* [1] if every open filter base in X has a cluster point. It is clear from the definition that every compact space is almost compact but not vice versa, as the reader may check from a counterexample in [1]. Another counterexample we will present also in this paper. The real line $\mathbb{R}$, if not otherwise specified, we consider as a topological space equipped with the natural, Euclidean topology, generated by all open intervals.

3 MAIN RESULT

Let us start with the following simple example. As we will show later, the existence of undominated strategies of both players is not a consequence of the classical Moulin's Theorem, but it follows from our generalization.

Example 3.1 Consider a normal form game of two players with the same sets of strategies $X_1 = X_2 = [0,1) \times \{0\} \cup \{1\} \times \{0,1,\ldots\}$. Let the corresponding utility functions of the players be

$$u_1(x_1,y_1) = \frac{x_1}{x_1+x_2} \cdot \left(\frac{8}{9}\right)^{y_2}, \;\; u_2(x_2,y_2) = \frac{x_2}{x_1+x_2} \cdot \left(\frac{9}{11}\right)^{y_1}.$$

It is easy to see that the pairs $(1,n) \in X_i$, where $n \in \{0,1,\ldots\}$ and $i = 1,2$, are equivalent, maximal and undominated strategies of the i-th player. $\square$

Theorem 3.2 *Let $G = (X_1, X_2, \dots, X_n, u_1, u_2, \dots, u_n)$ be a normal form game of n players. Suppose that for some $i \in \{1, 2, \dots, n\}$, X_i is almost compact and the utility function u_i is a continuous, real valued function of the argument $x_i \in X_i$. Then the i-th player has an undominated strategy.*

The proof of Theorem 3.2 will be presented during author's oral presentation. Now, let us demonstrate the utility of our generalization of Moulin's Theorem. Consider the game, described in the example. Although the utility functions u_i are continuous, the topology of X_i, induced from the real plane is not compact. For instance, the sequence $\{(1,n) | n = 0, 1, 2, \dots\}$ has no cluster point. Let us define another topology on X_i, where $i = 1, 2$, by the local base of a general point $(x, y) \in X_i$:

1. The point $(0,0)$ has neighborhoods of the form $[0, \varepsilon) \times \{0\}$, $0 < \varepsilon < 1$.
2. For every $x \in (0,1)$, the point $(x,0)$ has neighborhoods of the form $(x - \varepsilon, x + \varepsilon) \times \{0\}$, $0 < \varepsilon < \min\{x, 1 - x\}$.
3. For every $n = 0, 1, \dots$, the point $(1,n)$ has neighborhoods having the form $(1 - \varepsilon, 1) \times \{0\} \cup \{(1,n)\}$, where $0 < \varepsilon < 1$.

The new topology on X_i is now similar to the Euclidean topology on the unit segment $[0,1]$ but with one important difference – the right end point of the "segment" is present infinitely many times. The space X_i is T_1, but certainly non-Hausdorff and non-compact. Indeed, denoting $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$, the family $\{Y_n | n = 0, 1, \dots\}$ is an open cover of X_i, having no finite subcover. However, we can show that the new topology is almost compact. Let Ω be an open cover of X_i. The subspace $Y_0 = [0,1] \times \{0\} \subseteq X_i$ is compact since it is homeomorphous with the unit segment $[0,1]$, so there exists a finite subfamily $\{U_1, U_2, \dots, U_k\} \subseteq \Omega$ with $Y_0 \subseteq \bigcup_{j=1}^{k} U_j$. Then there is $r \in \{1, 2, \dots, k\}$ such that $(1,0) \in U_r$. But for every $n = 1, 2, \dots$ it follows $(1,n) \in \operatorname{cl} U_r$, so the closures of $\{U_1, U_2, \dots, U_k\}$ cover X_i, so X_i is almost compact. The utility functions u_i are continuous functions of the argument (x_i, y_i) since they are continuous on the open subspaces $Y_n = [0,1) \times \{0\} \cup \{(1,n)\}$ of X_i, $n = 0, 1, \dots$, homeomorphous to $[0,1]$. Hence, the existence of the undominated strategies now follows from Theorem 3.2.

4 CONCLUSION

Our previous considerations show that our generalization of Moulin's Theorem significantly extends the class of applicable tasks or problems. In addition they together with Example 3.1 demonstrate that non-Hausdorff and non-Euclidean topologies are really very natural, just from the real life.

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