SINGLE-CHANNEL QUEUING PROBLEMS APPROACH

Abdurrzzag TAMTAM, Doctoral Degree Programme (2)
Dept. of Telecommunications, FEEC, BUT
E-mail: xtamta00@stud.feec.vutbr.cz

Supervised by: Dr. Karol Molnár

ABSTRACT

The paper deals with one problem of single channel queuing problems. It shows that single-channel queuing model without losses can be used to solve queuing problems. It is composed of the queue and service center and uses the FIFO system. The station performs the service for the first element which appears in line.

1 INTRODUCTION

In order to describe queuing problems through mathematical formulation, some assumptions are made by considering arrivals and service as patterned by known function. Equations representing the distribution of the time between arrivals are used with other equations depicting other features such as the distribution of the service time. The relationship existing between these equations is the matter studied in waiting line theory. Arrivals of people or entry requirements (events) are customarily Poisson distributed. The duration of the service provided by people is usually exponentially distributed.

2 SINGLE-CHANNEL QUEUING PROBLEMS

Single-station or single-channel queuing problem is the name applied on those problems in which only one unit (station) is delivering the service as illustrated in Fig. 1, where circles represent the arrival elements (events) and a square represents a station which contains an element being serviced.

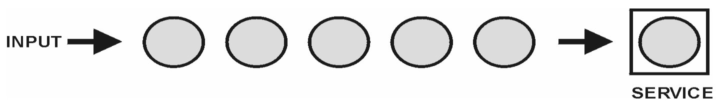

Fig. 1: *Single-channel queuing problem*

2.1 POISSON ARRIVALS

The Poisson is a discrete probability distribution and yields the number of arrivals in a given time. The exponential distribution is a continuous function and yields the distribution of the time intervals between arrivals. The Poisson distribution consider the behavior of arrivals as occurring at random and postulates the presence of a constant "λ" which is independent of the time. The constant λ represents the mean arrival rate or the number of arrivals per unit of time, and $\frac{1}{\lambda}$ is the length of the time interval between two consecutive arrivals. The Poisson distribution is expressed by the following formula: $P_k(T) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \qquad \begin{matrix} k = 1, 2, \ldots \\ 0 \le T < \infty \end{matrix}$

where the parameter λ is the probability of the arrival which occurs between the time t and $t+\Delta t$, and e is the base of the natural system of logarithms. The expected number of arrivals through the interval (0, T) is λT. We calculate the mean value μ of arrivals

$$\mu = \sum_{k=0}^{\infty} k P_k(T) = \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} e^{-\lambda T} = e^{-\lambda T} \lambda\ T \sum_{k=1}^{\infty} \frac{(\lambda\ T)^k}{k!} = e^{-\lambda T} \lambda\ t\, e^{\lambda T} = \lambda\ T\ .$$

By assigning the value 1 to the period i.e. T = 1, we have $P_k = \frac{\lambda}{k!} e^{-\lambda} \qquad k - 1, 2, \ldots$

The probability that per the interval (0,T) does not come any event is $P_0(T) = e^{-\lambda T}$

The complementary situation is described as follows: $P_{k \ge 1}(T) = \sum P_k(T) = 1 - e^{-\lambda T}$

The difference in the last formula is the distribution function of the exponential distribution with the function density equal to $\lambda\, e^{-\lambda T}$, so the mathematical expression of the distribution function is then $F(T) = P(\tau \le T) = \begin{cases} 0 & T < 0 \\ 1 - e^{-\lambda T} & T \ge 0 \end{cases}$

The function $F(T)$ has this sense: It is the probability that the time interval between two consecutive arrivals will be equal or less than the value of T. We calculate the mean time between two arrivals. $E(\tau_0) = \int_0^{\infty} T\, dF(T) = \int T\lambda\ e^{-\lambda T}\ dT = -\frac{1}{\lambda}\left[e^{-\lambda T}\right]_0^{\infty} = \frac{1}{\lambda}$

2.2 EXPONENTIAL SERVICE TIMES

When the servicing of a unit takes place between time t and Δt (for Δt sufficiently small) the service times are given by exponential distribution, while the service rates are given by the Poisson distribution. The parameter $\mu\ T$ indicates that μ is a constant of proportionality, which is independent of time, of the queue length, or of the features. Again, calling k the number of potential services which can be performed in the interval (0, T) ,the Poisson formula for the servicing rate is $p_k(T) = \frac{(\mu T)}{k!} e^{-\mu T} \qquad \begin{matrix} k = 1, 2, \ldots \\ 0 \le T < \infty \end{matrix}$

The mean servicing rate which is the expected number of services performed in one unit of time) is indicated by μ when the servicing time is exponential. It can be found approximately by dividing the output (time) of the services delivered along the period T. by the portion of T in which the services are really operating. The mean servicing time MST is the reciprocal of μ, $MST = \frac{1}{\mu}$

2.3 SINGLE-CHANNEL QUEUING MODEL WITHOUT LOSSES (M/M/1)

Customarily, the inputs, as well as the length of time required by the station to perform the requested work, are considered to arrive at random. The servicing rate is independent of the number of elements in line. The station performs the service for the first element which appears in line (FIFO System – First In, First Out). When the service is busy, the incoming element waits in line in order of arrival until the previous element leaves the channel at the end of its service. We suppose an infinite source of arrival elements. This system is often called the system of bulk service (SBS). It is composed of the queue and service center.

We put a list of notations which will be useful in the next mathematical considerations:

λ = mean arrival rate (number of arrivals per unit of time)

μ = mean service rate (per channel)

n = number of elements in SBS

$\lambda\Delta t$ = probability that a new element enters the SBS between t and $t + \Delta t$ time interval

$\mu\Delta t$ = probability that an element has received service (completely finished) between t and $t + \Delta t$ time interval

$1-\lambda\Delta t$ = probability of having no arrivals in the interval $(t, t + \Delta t)$

$1-\mu\Delta t$ = probability of having no elements serviced during the interval $(t, t + \Delta t)$

$P_{n+1}(t)$ = probability of having n +1 elements in the SBS system at time t

$P_{n-1}(t)$ = probability of having n - 1elements in the SBS system at time t

$P_n(t + \Delta t)$ = probability of having n elements in the SBS system at time t

Let us suppose $n > 0$. We calculate the probability $P_n(t + \Delta t)$

$$P_n(t + \Delta t) = P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t) + P_{n+1}(t)(\mu\Delta t)(1-\lambda\Delta t) + P_{n-1}(t)(\lambda\Delta t)(1-\mu\Delta t) + +P_n(t)(\lambda\Delta t)(\mu\Delta t) \qquad \text{for } n > 0.. \qquad (1)$$

Δt is a very small interval hence we can omit its square i.e. we approximate it by zero. We receive under given supposed condition from (1) the following system of equations:

$$P_n(t + \Delta t) = P_n(t) + (\mu\Delta t)P_{n+1}(t) + (\lambda\Delta t)P_{n+1}(t) - ((\lambda + \mu)\Delta t)P_n(t) \qquad n > 0$$

$$P_n(t + \Delta t) - P_n(t) = (\mu\Delta t)P_{n+1}(t) + (\lambda\Delta t)P_{n+1}(t) - ((\lambda + \mu)\Delta t)P_n(t) \qquad n > 0$$

we divide the equation by Δt and hence

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \mu P_{n+1}(t) + \lambda P_{n+1}(t) - (\lambda + \mu)P_n(t) \qquad (2)$$

when Δt approaches zero, the following differential equations can be stated:

$$\frac{dP_n(t)}{dt} = \mu P_{n+1}(t) + \lambda P_{n+1}(t) - (\lambda + \mu)P_n(t) \qquad (3)$$

which expresses the relationship among the probabilities P_n , P_{n-1} , P_{n+1} at the time t and the mean arrival rate (λ) and the mean service rate (μ).

The probability that no elements will be in the (SBS) is given by the equation (3).

$$\frac{d\,P_0(t)}{d\,t} = -\lambda P_0(t) + \mu P_1(t) \qquad (4)$$

We suppose that the system is working for an unbounded time interval and that it pass to the steady state-condition. In this moment $P_n(t)$, which are probabilities which were be dependent on time *t* they become independent according to time and thus

$$\frac{d\,P_n}{d\,t} = 0 \qquad \text{for } n = 0, 1, 2, 3, \ldots \qquad (5)$$

Thus (3) and (4) are transformed on homogenous linear equations :

$$0 = \mu P_{n+1} + \lambda P_{n-1} - (\lambda + \mu) P_n \qquad n = 1, 2, 3, \ldots \qquad (6)$$

$$0 = -\lambda P_0 + \mu P_1 \,, \qquad (7)$$

where $P_1, P_2, P_3, \ldots, P_n, \ldots$ are unknown values. We have from (7)

$P_1 = \frac{\lambda}{\mu} P_0$ and after a rewriting of the equation (6) for $n = 1$ we have

$0 = \mu P_2 + \lambda P_0 - (\lambda + \mu) P_1$

Hence $\mu P_2 = (\lambda + \mu) P_1 - \lambda P_0$

and after the substitution for P_0 we have : $P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$

hence using the induction we receive the common formula for the probability that the single-channel system contains together just *n* (in the queue – in the line and in the service center) arrived elements (demands).

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad (8)$$

We denote the ratio $\frac{\lambda}{\mu}$ by ρ. We call it traffic intensity, which is the expected service per unit of time measured in erlangs (in honor of A.K.Erlang, who is considered the father of the queuing theory). For next we suppose that $\rho < 1$. Now we use the fact that $\sum_{n=0}^{\infty} P_n = 1$

and we substitute P_n. $\sum_{n=0}^{\infty} P_n = P_0 \left(1 + \rho + \rho^2 + \ldots + \rho^n + \ldots\right) = 1$

hence $P_0 = \frac{1}{1 + \rho + \rho^2 + \ldots} = \left[\sum_{n=0}^{\infty} \rho^n\right]^{-1}$

which is the geometric progression with the quotient $\rho < 1$, hence $\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$.

We have for individual cases: $P_0 = 1 - \rho \qquad (9)$

$$P_n = (1-\rho)\rho^n \qquad n = 1, 2, 3, \ldots \qquad (10)$$

The equation (10) expresses the probability of existence of waiting queue of the length $n - 1$ for $n > 0$. (Note that this equation is valid, as already indicated, only when $\lambda < \mu$).

The average number of elements (events), both waiting in the queue and attended in service is: $m = \sum_{n=0}^{\infty} n\,\mathrm{P}_n = \sum n(1-\rho)\rho^n = (1-\rho)\rho\sum n\rho^{n-1} = (1-\rho)\rho\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho}$,

after rearrangements we obtain $m = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$.

The infinite series is of exponential type. Hence we can apply the following rule:

$\int S(\rho)d\rho = \sum_{n=1}^{\infty} n\rho^{n-1}d\rho$, using this formula we receive: $\int S(\rho)d\rho = \sum_{n=1}^{\infty}\rho^n = \frac{\rho}{1-\rho}$

But we need the value of S(ρ) which is the derivative of the fraction $\frac{\rho}{1-\rho}$.

The mean length (m_Q) of the queue (of the waiting elements excluding the element under the service process is obtained from the definition of the mean value in theory of probability and we have:

$$m_Q = \sum_{n=1}^{\infty}(n-1)\mathrm{P}_n = \sum_{n=1}^{\infty} n\,\mathrm{P}_n - \sum_{n=1}^{\infty}\mathrm{P}_n = m - (1-\mathrm{P}_0) = m - \rho = \frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\lambda^2}{\mu-\lambda} .$$

The mean time between arrivals (h), where the arrivals are Poisson distributed, is obtained by reciprocating the mean arrival rate λ, hence $h = \frac{1}{\lambda}$.

The average time of demurrage of the element in all the system, i.e. in queue and in service (WTS) is expressed in terms of λ and μ as follows: $WTS = \frac{m}{\lambda} = \frac{1}{\lambda}\frac{\lambda}{\mu-\lambda} = \frac{1}{\mu-\lambda}$.

The average waiting time of an element in queue (WTQ) is expressed in terms of λ and μ as follows: $WTQ = \frac{m_Q}{\lambda} = \frac{1}{\lambda}\frac{1}{\mu}\frac{\lambda^2}{\mu-\lambda} = \frac{\lambda}{\mu}\frac{1}{\mu-\lambda} = \frac{\rho}{\mu(1-\rho)}$

3 CONCLUSIONS

The former model admits the queues of arbitrary lengths. Equation (10) gives the probability that an element will not have to wait at all upon its arrival at the service station before going into one-channel service. A model with queues of arbitrary length is often called a model without losses.

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