METHOD OF GENERALIZED EIGENVECTORS IN LINEAR DISCRETE SYSTEMS WITH CONSTANT COEFFICIENTS

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ABSTRACT

A new construction of the fundamental solution system of the system of homogeneous linear difference equations with constant coefficients is presented in this article. The method of the generalized eigenvectors is used and the difference between the construction of the fundamental solution systems for the difference equations and differential equations is shown.

1 INTRODUCTION

Consider the system of linear difference equations with constant coefficients

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n), \tag{1}$$

where $\mathbf{A}$ is a real square matrix of the m-th order and $\mathbf{x} = [x_1, x_2, \ldots, x_m]^T$ is the desired vector of the solution, which depends on the variable n, it means $\mathbf{x} = \mathbf{x}(n)$, $n \geq n_0$, $n_0 \in \mathbb{Z}$. Assume that the non-zero solution $\mathbf{x}(n) \not\equiv \mathbf{0}$ of the system (1) has the form

$$\mathbf{x}(n) = \mathbf{v}\lambda^n, \tag{2}$$

where λ is a suitable complex number and $\mathbf{v}$ is an eligible constant vector. By substituting (2) into the system (1) we get

$$\mathbf{v}\lambda^{n+1} = \mathbf{A}\mathbf{v}\lambda^n$$

and after the modification

$$\mathbf{A}\mathbf{v}\lambda^n - \lambda\mathbf{v}\lambda^n = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}\lambda^n = \mathbf{0},$$

where $\mathbf{I}$ is the $m \times m$ unit matrix and $\mathbf{0}$ is the zero vector. From the last relation (after dividing by λ^n) we get

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}. \tag{3}$$

The system (3) does not already contain the independent variable n, it is the system of linear algebraic equations with respect to the components of the vector $\mathbf{v}$. It will have the non-zero solution $\mathbf{v}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{4}$$

The equation (4) is called the characteristic equation. The roots of characteristic equation are called the eigenvalues of the matrix $\mathbf{A}$. If $\lambda = \lambda^*$ is the eigenvalue of the matrix $\mathbf{A}$ and the vector $\mathbf{v} = \mathbf{v}^*$ is the non-zero solution of the system

$$(\mathbf{A} - \lambda^*\mathbf{I})\mathbf{v} = \mathbf{0},$$

then the vector $\mathbf{v}^*$ is called the eigenvector of the matrix $\mathbf{A}$. The vector function

$$\mathbf{x}(n) = \mathbf{v}^*(\lambda^*)^n$$

is one of the solutions of the system (1).

2 THE CASE OF THE DIFFERENT ROOTS OF THE CHARACTERISTIC EQUATION

In case when the characteristic equation (4) has m respectively various real roots

$$\lambda_1, \lambda_2, \ldots, \lambda_m$$

and m corresponding (non-zero) eigenvectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m,$$

the construction of the general solution of the system (1) is simple and the system of the vector functions

$$\mathbf{v}_1\lambda_1^n, \mathbf{v}_2\lambda_2^n, \ldots, \mathbf{v}_m\lambda_m^n$$

forms the fundamental solution system of the system (1). One may express the general solution of this system in the form

$$\mathbf{x}(n) = K_1\mathbf{v}_1\lambda_1^n + K_2\mathbf{v}_2\lambda_2^n + \ldots + K_m\mathbf{v}_m\lambda_m^n,$$

where $K_1, K_2, \ldots, K_m$ are arbitrary constants.

3 THE MULTIPLE ROOTS OF THE CHARACTERISTIC EQUATION

The construction of the fundamental solution system is more complicated in the case, when the roots of the characteristic equation are multiple. In this case the eigenvalue λ with the multiplicity s generates s corresponding linear independent solutions, $s \in \{2, 3, \ldots, m\}$. One of them has the form we supposed, it means the form (2).

If the eigenvalue is a complex number, $\lambda = \alpha + \mathrm{j}\beta$, where j is the imaginary unit and

$\alpha, \beta \in \mathbb{R}$, the complex conjugate number $\bar{\lambda} = \alpha - j\beta$ will be the eigenvalue with the same multiplicity too. If the complex solution $\mathbf{x} = \mathbf{x}(n)$ of the system (1) exists, two real solutions $\mathbf{x}_1(n)$, $\mathbf{x}_2(n)$ can be obtained as

$$\begin{aligned} \mathbf{x}_1(n) &= \operatorname{Re}\mathbf{x}(n) = \operatorname{Re}[\mathbf{v}\lambda^n], \\ \mathbf{x}_2(n) &= \operatorname{Im}\mathbf{x}(n) = \operatorname{Im}[\mathbf{v}\lambda^n], \end{aligned}$$

where Re means the real part and Im the imaginary part of the complex solution $\mathbf{x}(n)$.

3.1 THE CONTINUOUS SYSTEM - THE SYSTEM OF DIFFERENTIAL EQUATIONS

The system of differential equations presents the analogy of our problem. Therefore, before the explanation of the case considered in the case of difference system, we describe shortly the similar situation for the ordinary differential systems. We consider the system of linear differential equations with constant coefficients

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$$

and suppose that the corresponding characteristic equation has the root λ with the multiplicity s. Assume that the rank $(\mathbf{A} - \lambda\mathbf{I}) = m - 1$. Then the vector functions

$$\mathbf{v}_1 e^{\lambda t},\ (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t},\ \ldots,\ \left(\mathbf{v}_1 \cdot \frac{t^{s-1}}{(s-1)!} + \mathbf{v}_2 \cdot \frac{t^{s-2}}{(s-2)!} + \ldots + \mathbf{v}_{s-1} t + \mathbf{v}_s\right) e^{\lambda t},$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are non-zero vectors satisfying the systems (see [2])

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}, \tag{5}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1, \tag{6}$$

$$\vdots$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_s = \mathbf{v}_{s-1}, \tag{7}$$

creates the fundamental solution system.

3.2 THE DISCRETE SYSTEM

The construction of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_s$ in the case of the system of difference equations is different from the construction of the vectors in the continuous case. These vectors cannot be found as the non-zero solutions of the systems (5)–(7) and it is necessary to use different systems.

Let us now focus our attention on the system (1). Recall that the relation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

holds, where $\mathbf{v}_1$ is the eigenvector of the matrix $\mathbf{A}$ corresponding to the eigenvalue λ. The vector

$$\mathbf{x}_1(n) = \mathbf{v}_1 \lambda^n$$

is the solution of the system (1). Assume that the rank $(\mathbf{A}-\lambda\mathbf{I}) = m-1$ and the multiplicity of the root $s=2$. Let us define the vector

$$\mathbf{x}_2(n) = (\mathbf{v}_1 n + \mathbf{v}_2)\lambda^n, \tag{8}$$

where $\mathbf{v}_2$ is a suitable non-zero vector. By substituting the vector (8) into the system (1) we get

$$\Big[\mathbf{v}_1(n+1) + \mathbf{v}_2\Big]\lambda^{n+1} = \mathbf{A}(\mathbf{v}_1 n + \mathbf{v}_2)\lambda^n. \tag{9}$$

The equation (9) can be divided by the term λ^n. Comparing the coefficients of the identical functional terms we obtain

$$\begin{array}{lrcl} n^1: & \mathbf{v}_1\lambda & = & \mathbf{A}\mathbf{v}_1, \\ n^0: & (\mathbf{v}_1+\mathbf{v}_2)\lambda & = & \mathbf{A}\mathbf{v}_2. \end{array}$$

After the simplification we have two systems

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}, \tag{10}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \lambda\mathbf{v}_1, \tag{11}$$

which determine the relations between the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, if the matrix $\mathbf{A}$ has the eigenvalue λ with the multiplicity $s=2$, two linear independent solutions are

$$\begin{array}{rcl} \mathbf{x}_1(n) & = & \mathbf{v}_1\lambda^n, \\ \mathbf{x}_2(n) & = & (\mathbf{v}_1 n + \mathbf{v}_2)\lambda^n. \end{array}$$

Next, we assume that the multiplicity of the root $s=3$. Let us define the vector

$$\mathbf{x}_3(n) = \left(\mathbf{v}_1\frac{n^2}{2} + \mathbf{v}_2 n + \mathbf{v}_3\right)\lambda^n, \tag{12}$$

where $\mathbf{v}_3$ is a suitable non-zero vector. If we substitute the vector (12) into the system (1), we get

$$\left[\mathbf{v}_1\frac{(n+1)^2}{2} + \mathbf{v}_2(n+1) + \mathbf{v}_3\right]\lambda^{n+1} = \mathbf{A}\left(\mathbf{v}_1\frac{n^2}{2} + \mathbf{v}_2 n + \mathbf{v}_3\right)\lambda^n. \tag{13}$$

After the division of the equation (13) by the term λ^n and comparision the coefficients of the identical functional terms we obtain

$$\begin{array}{lrcl} n^2: & \frac{1}{2}\mathbf{v}_1\lambda & = & \frac{1}{2}\mathbf{A}\mathbf{v}_1, \\ n^1: & (\mathbf{v}_1+\mathbf{v}_2)\lambda & = & \mathbf{A}\mathbf{v}_2, \\ n^0: & \left(\frac{1}{2}\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3\right)\lambda & = & \mathbf{A}\mathbf{v}_3. \end{array}$$

If we simplify these equations, we obtain three systems

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}, \tag{14}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \lambda\mathbf{v}_1, \tag{15}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 = \lambda\left(\frac{1}{2}\mathbf{v}_1 + \mathbf{v}_2\right), \tag{16}$$

which determine the relations between the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. If the matrix $\mathbf{A}$ has the eigenvalue λ with the multiplicity $s = 3$, three linear independent solutions are

$$\begin{aligned}
\mathbf{x}_1(n) &= \mathbf{v}_1\lambda^n, \\
\mathbf{x}_2(n) &= (\mathbf{v}_1 n + \mathbf{v}_2)\lambda^n, \\
\mathbf{x}_3(n) &= \left(\tfrac{1}{2}\mathbf{v}_1 n^2 + \mathbf{v}_2 n + \mathbf{v}_3\right)\lambda^n.
\end{aligned}$$

It is obvious that the systems (10), (11), in case of the root λ with the multiplicity $s = 2$, are included in the systems (14)–(16), where the root λ is triple. According to this conclusion, one may construct the solution corresponding to the root λ with the multiplicity s. Then, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are non-zero vectors satisfying the systems

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}, \tag{17}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \lambda\mathbf{v}_1, \tag{18}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \lambda\left(\frac{1}{2}\mathbf{v}_1 + \mathbf{v}_2\right), \tag{19}$$

$$\vdots$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_s = \lambda\left(\frac{\mathbf{v}_1}{(s-1)!} + \frac{\mathbf{v}_2}{(s-2)!} + \ldots + \frac{\mathbf{v}_{s-2}}{2!} + \mathbf{v}_{s-1}\right), \tag{20}$$

the vector functions

$$\mathbf{v}_1\lambda^n,\ (\mathbf{v}_1 n + \mathbf{v}_2)\lambda^n,\ \ldots,\ \left(\mathbf{v}_1 \cdot \frac{n^{s-1}}{(s-1)!} + \mathbf{v}_2 \cdot \frac{n^{s-2}}{(s-2)!} + \ldots + \mathbf{v}_{s-1} n + \mathbf{v}_s\right)\lambda^n,$$

creates the fundamental solution system of (1).

4 CONCLUSION

The permanent expansion and usage of the discrete systems in technique force us to look for the new ways, how to describe these systems mathematically. The way with the aid of difference equations is often used, which represents the analogy of differential equations with the continuous systems in a certain way. In the contribution, a new approach has been developed for the construction of the fundamental solution system of the discrete equation system with the aid of generalized discrete eigenvectors. This approach is different from the analogical continuous case. The systems (5)–(7) and (17)–(20) represent the main discrepancy. It seems that the developed algorithm is suitable for creating of a new software tool.

REFERENCES

[1] Elaydi, S. N.: An Introduction to Difference Equations, Second Edition, Springer, 1999.

[2] Kuben, J.: Obyčejné diferenciální rovnice, University of Defense in Brno, Brno, 2000.

[3] Pták, P.: Diferenciální rovnice, Laplaceova transformace, Czech Technical University in Prague, Prague, 1999.