OPTIMIZING OF QUANTUM STATE DETECTOR

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ABSTRACT

In the article we are interested in quantum optical communication. Hence the information is carried with using photons (generally with states of systems). We concentrate our attention on the quantum description of states which are prepared by light sources. And consequently we calculate bit error rate for various types of transmissions. If the measuring process is acted in the special basis, the minimum bit error rate is obtained.

1 INTRODUCTION

Firstly, we introduce some basic notions used in quantum communication theory. In the communication structure there are two the most important phases. The first phase is the quantum system preparation (this phase is represented by light sources). And the second phase, analogous to the first one, is the measuring process (light detector). Hence preparations and measurements are the primitive notions of quantum theory.

A quantum system is some kind of abstraction which exists in the mind of the observer and is given by particular measuring process. In the other words, a quantum system is something that admits a quantum description in a given test. A definition of quantum state has a clearer meaning. A state is fully characterized by the probabilities of the various outcomes of possible measurements.

Mathematically the pure states of systems are represented by state vectors. These vectors are components of complex vector space (its dimension depends on the number of maximum outcomes of test). Every physical process (every observable) is completely determined by the particular matrix. Each state after measurement of observable $\mathbf{A}$ can be written as a superposition of the eigenstates of $\mathbf{A}$. Thus these eigenstates are given by a particular maximal test (roughly speaking, maximal test is a measuring process with ideal resolution and with maximum number of different outcomes).

There are two types of states. Pure states and mixed states. If a given system has defined probabilities for each outcome in a maximal test then the system is in a pure state. If following is untrue, it is said that the system is in a mixed state.

2 *N*-STATE DEMODULATION

Suppose N-state transmission of information. Each state is represented by a different pure state. For instance, if two-state modulation is used, 0 bit can be represented by horizontally polarized photon and 1 bit by vertically polarized photon. Detection space in the receiver side can be divided into N disjoint subspaces $\{N_j\}$. Each subspace is characterized by a particular projection operator $\hat{P}_j$ [1]

$$\hat{P}_j = \sum_{k\left(b_k \in \{N_j\}\right)} |b_k\rangle\langle b_k| \tag{1}$$

where $|b_k\rangle$ is an eigenvector of observable which is measured. For example, an observable can test polarization of photon. We noticed, that n-th decision is represented by n-th pure state. But in general, several pure states belong to the n-th subspace. For instance, horizontally polarized photon was disturbed by channel and there is a little probability that the photon will passed the test on the vertically polarization. But it still represents 0 bit. Hence n-th subspace is represented by a mixed state. Each mixed state can be written as [1]

$$\hat{W} = \sum_i w_i |\varphi_i\rangle\langle\varphi_i| \tag{2}$$

where $\hat{W}$ denotes density matrix and w_i is the probability that the state $|\varphi_i\rangle$ occurs in given mixed state (state $|\varphi_i\rangle$ is a superposition of eigenvectors $|b_i\rangle$). Note, that $\langle b_i | b_j\rangle$ denotes scalar product and physical meaning of $\left|\langle b_i | b_j\rangle^2\right|$ is probability that pure state $|b_j\rangle$ passes the test for $|b_i\rangle$ (that means if pure state $|b_i\rangle$ is tested then it certainly passes the test). A useful meaning of density matrixes is given in [3]

The average probability of correct decision is

$$P_C = \sum_n^N p_n \sum_{k\left(b_k \in \{N_j\}\right)} \langle b_k | \hat{W}_n | b_k\rangle = \sum_n^N p_n \hat{W}_n \hat{P}_n \tag{3}$$

where p_n denotes the probability that $\hat{W}_n$ occurs. Eq. (3) can be written as

$$P_C = \sum_n^N p_n \mathrm{Tr}\left(\hat{W}_n \hat{P}_n\right) \tag{4}$$

where $\mathrm{Tr}(\cdot)$ denotes trace of the matrix. The average probability of error is then expressed as $P_E = 1 - P_C$.

3 MINIMIZE OF ERROR PROBABILITY

For simplicity we assume only two detection subspaces (for instance, OOK). Then from Eq. (3) one obtains

$$P_C = p_1 \mathrm{Tr}\left(\hat{W}_1 \hat{P}_1\right) + \left(1 - p_1\right)\left(\mathrm{Tr}\hat{W}_2 - \mathrm{Tr}\hat{W}_2 \hat{P}_1\right). \tag{5}$$

From Eq. (5) it can be seen that to achieve minimal average probability of correct decision the term $\mathrm{Tr}\left(\Delta\hat{W}\hat{P}_1\right)$ must be maximized ($\Delta\hat{W} = \hat{W}_1 - \hat{W}_2$). Now, suppose that operator $\Delta\hat{W}$ satisfies equation (it is not restriction)

$$\Delta\hat{W}\left|\alpha_l\right\rangle = \alpha_l\left|\alpha_l\right\rangle, \quad \left\langle\alpha_l \middle| \alpha_m\right\rangle = \delta_{l,m} \tag{6}$$

where $\left|\alpha_l\right\rangle$ are eigenvectors of $\Delta\hat{W}$ a α_l are its eigenvalues. Then Eq. (6) can be rewritten as

$$\mathrm{Tr}\left(\Delta\hat{W}\hat{P}_1\right) = \sum_{k\left(b_k\in N_j\right)} \left\langle b_k\right| \Delta\hat{W}\sum_l \left|\alpha_l\right\rangle\left\langle\alpha_l \middle| b_k\right\rangle = \sum_l \sum_{k\left(b_k\in N_j\right)} \alpha_l \left|\left\langle\alpha_l \middle| b_k\right\rangle\right|^2. \tag{7}$$

Eigenvalues α_l of operator $\Delta\hat{W}$ are real numbers. To obtain minimal error probability, positive numbers α_l achieve the maximal value and negative α_l achieve the minimal value. The probability term $\left|\left\langle\alpha_l \middle| b_k\right\rangle\right|^2$ occurs boundary values 1 and 0 if and only if $\left\langle\alpha_l \middle| b_k\right\rangle = \delta_{lk}$. Hence for obtaining the minimal error probability projection operator $\hat{P}_1$ is expressed as follow

$$\hat{P}_{1\min} = \sum_{k\left(\alpha_k\geq 0\right)} \left|\alpha_k\right\rangle\left\langle\alpha_k\right|. \tag{8}$$

The physical interpretation of the result (8) is following. Instead of using test which is represented by Eq. (1), there is used a different test determined by eigenvectors $\left|\alpha_l\right\rangle$. Using previous equations the resulting formula is obtained ($p_1 = 0{,}5$ was supposed)

$$P_{\min E} = \frac{1}{2}\left(1 - \sum_{l\left(\alpha_l\geq 0\right)} \left\langle\alpha_l\right|\Delta\hat{W}\left|\alpha_l\right\rangle\right) = \frac{1}{2}\left(1 - \sum_{l\left(\alpha_l\geq 0\right)} \alpha_l\right). \tag{9}$$

Suppose that pure state of photon $\left|\varphi_1\right\rangle$ represents 0 bit and pure state $\left|\varphi_2\right\rangle$ represents 1 bit. Because a photon is a particle which is described by two dimension complex vector space (of course, only when polarization is considered), states $\left|\varphi_1\right\rangle, \left|\varphi_1\right\rangle$ can be expressed as

$$\left|\varphi_1\right\rangle = a_{11}\left|b_1\right\rangle + a_{21}\left|b_2\right\rangle, \quad \left|\varphi_2\right\rangle = a_{12}\left|b_1\right\rangle + a_{22}\left|b_2\right\rangle. \tag{10}$$

To obtain $P_{\min E}$ Eq. (6) must be solved. The result is $\alpha_{1,2} = \pm\sqrt{1 - \left|\left\langle\varphi_1 \middle| \varphi_2\right\rangle\right|^2}$. Hence (with using Eq. (9))

$$P_{\min E} = \frac{1}{2}\left(1 - \sqrt{1 - \left|\left\langle\varphi_1 \middle| \varphi_2\right\rangle\right|^2}\right). \tag{11}$$

And the measuring process is characterized by new orthogonal vectors

$$\left|\alpha_1\right\rangle = c_{11}\left|b_1\right\rangle + c_{21}\left|b_2\right\rangle, \quad \left|\alpha_2\right\rangle = c_{12}\left|b_1\right\rangle + c_{22}\left|b_2\right\rangle \tag{12}$$

where $c_{2j} = \dfrac{\left(a_{21}a_{11}^* - a_{22}a_{12}^*\right)c_{1j}}{\alpha_j - \left|a_{21}\right|^2 + \left|a_{22}\right|^2}$, $j = 1, 2$. Coefficients c_{1j} are chosen in such a way, that states in Eq. (12) are normalized. In Fig. (1) there is calculated error probability for this transmission.

Let us denote $|n\rangle$ as a number state. Hence $|n\rangle$ is the state which contains exactly n identical photons. Normalized coherent states are given by the following superposition of number states [1]

$$|\kappa\rangle = \mathrm{e}^{-\frac{1}{2}|\kappa|^2} \sum_{n=0}^{\infty} \frac{\kappa^n}{\sqrt{n!}} |n\rangle, \tag{13}$$

where $\kappa = \sqrt{\bar{n}}\mathrm{e}^{\mathrm{i}\delta}$ and $\bar{n}$ is the average number of photons in state $|\kappa\rangle$. Eq. (13) describes laser radiation. Thus photons in a laser pulse have the same wavelength and polarization. So these parameters do not occur in the Eq. (13). Let us suppose OOK. 1 bit is represented by state $|\kappa\rangle$ and 0 bit by state $|0\rangle$. With using Eq. (4) we can write

$$P_C = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\langle n|\kappa\rangle\langle\kappa|n\rangle}{\langle\kappa|\kappa\rangle} + \frac{1}{2}\langle 0|0\rangle\langle 0|0\rangle = \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{\bar{n}^n}{n!}\mathrm{e}^{-\bar{n}} + 1\right) = \frac{1}{2}\left(2 - \mathrm{e}^{-\bar{n}}\right) \tag{14}$$

Error probability is $P_E = \frac{1}{2}\mathrm{e}^{-\bar{n}}$. And finally minimal error probability is given by $P_{\min E} = \frac{1}{2}\left(1 - \sqrt{1 - \mathrm{e}^{-\bar{n}}}\right)$. These two probabilities are shown in Fig. (2).

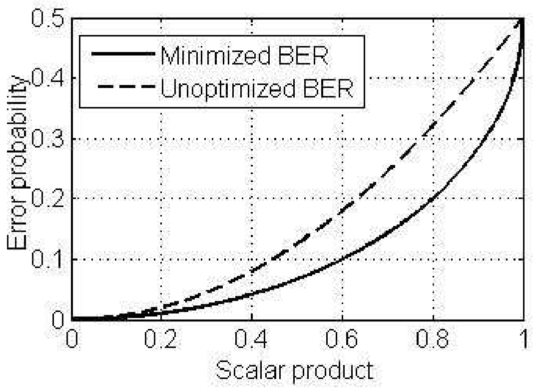

Fig. 1: *Single photon transmission*

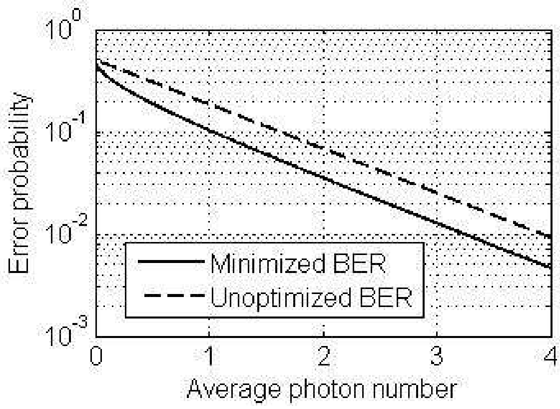

Fig. 2: *Coherent states*

4 A PHOTON IN A COHERENT STATE

In the previous section about coherent states the polarization of photon was not considered. But if a single-photon transmission is considered then polarization of photons play a key role. The right-handed photon (different polarization can be used) prepared by a laser diode is represented by the following state ($\bar{n} = 1$)

$$|R\rangle = \mathrm{e}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\delta n}}{\sqrt{n!}} |n, r\rangle. \tag{15}$$

A left-handed photon is given in the similar way (symbols R, r are replaced by L, l). For a single-photon transmission is important only situation when exactly one photon is detected. For this, Eq. (15) is rewritten as (the phase factor δ is placed zero, because we are interested in probabilities)

$$|R\rangle = \mathrm{e}^{-0,5}|1, r\rangle + \left(1-\mathrm{e}^{-1}\right)^{0,5}|n \neq 1, r\rangle \tag{16}$$

where $|n \neq 1, r\rangle$ denotes the states which contain $0, 2, \ldots, \infty$ right-handed photons. An arbitrary polarization state of photon can be expressed as (a right-handed photon satisfies conditions $a_1 = 1, a_2 = 0$)

$$\begin{aligned}|\varphi\rangle = a_1|R\rangle + a_2|L\rangle = a_1\mathrm{e}^{-0,5}|1, r\rangle + a_1\left(1-\mathrm{e}^{-1}\right)^{0,5}|n \neq 1, r\rangle \\ + a_2\mathrm{e}^{-0,5}|1, l\rangle + a_2\left(1-\mathrm{e}^{-1}\right)^{0,5}|n \neq 1, l\rangle\end{aligned} \tag{17}$$

Coefficients a_1, a_2 characterize the light source and can be evaluated by virtue of statistical measuring of $|\varphi\rangle$. The minimal bit error rate of two-state modulation is given by (11). States $|\varphi_1\rangle, |\varphi_2\rangle$ are given by the first part of relation (17), hence $|R\rangle, |L\rangle$ representation is used. While states $|\varphi_1\rangle, |\varphi_2\rangle$ are expressed in basis $|n, r\rangle, |n, l\rangle$ ($n = 0, 1, ..., \infty$) and then are put in formula (11), the higher BER is occurred. That is why the detector also detects a number of photon different from 1. These transmissions contain no information. But this kind of BER is different from the one given by equation (11) and can be eliminated by increasing of bit rate. Hence information is obtained when the detector detects exactly one arbitrary polarized photon. From (16) can be seen that bit rate is reduce by factor e^{-1}. In the other words, in average three laser pulses are needed for successful transmission of a classical bit.

The states of basis $|n, r\rangle, |n, l\rangle$ are simultaneously eigenstates of the energy operator and one component of moment of momentum operator. That is why they form an orthogonal set.

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