PIEZOCERAMIC SENSOR: BASIC NOTES FOR FINITE ELEMENT MODEL

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ABSTRACT

The positive reasons to use simulation of piezoelectric devices, especially finite element method (FEM), are presented. Basic ideas of finite element method (FEM) for solving the problem of anisotropic piezoelectric media are mentioned. The simulation as an axisymmetric task is mentioned as well.

1 INTRODUCTION

Piezoelectric materials are widely used in electromechanical sensors and actuators such as robotic sensors, ultrasonic transducers for medical imaging and nondestructive testing (NDT). In the past, the development of electroacoustic transducers was primarily based on trial and error, which is time-consuming and therefore expensive. In computer age, computational power can be employed. The main purposes of computer simulations in transducer development are [1]:

- optimization of transducer design without time-consuming experiments,
- deeper insight into the wave propagation in piezoelectric solids.

The models commonly used to simulate the mechanical and electrical behavior of piezoelectric transducers generally introduce simplifying assumptions that are often invalid for actual design. The geometries of practical transducer are often two dimensional (2D) or three dimensional (3D). The most popular models, such as Manson’s model or the KLM model are only one dimensional (1D). For the 2D and 3D simulation of piezoelectric media the complete set of fundamental equations governing piezoelectric media has to be solved. The finite difference or finite element are however sufficiently general to handle these differential equations. The finite element method was preferred because it is capable of handling complex geometries.

2 THEORY

The matrix equations relating mechanical and electrical quantities in piezoelectric media are the basis for the derivation of the finite element model [1, 2]:

$$\begin{aligned} \mathbf{T} &= \mathbf{c}^E \mathbf{S} - \mathbf{e}^T\, \mathbf{E} \\ \mathbf{D} &= \mathbf{e}\, \mathbf{S} + \varepsilon^S\, \mathbf{E} \end{aligned} \qquad (1)$$

T vector of mechanical stresses
S vector of mechanical strains
E vector of electric field
D vector of dielectric displacement
$\mathbf{c}^E$ mechanical stiffness matrix for constant electric field *E*
ε^S permittivity matrix for constant mechanical strain *S*
e piezoelectric matrix, $[e]^T$ transposed

The electric field **E** is related to electrical potential ϕ by

$$\mathbf{E} = -grad\,\phi \qquad (2)$$

and the mechanical strain S to the mechanical displacement u in the Cartesian coordinates by

$$\mathbf{S} = \begin{vmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{vmatrix}^T \mathbf{u} = \mathbf{B}\,\mathbf{u} \qquad (3)$$

The elastic behavior of piezoelectric media is governed by Newton's law:

$$div\mathbf{T} = \rho \partial^2 \mathbf{u} / \partial t^2 \qquad (4)$$

where ρ is density of piezoelectric medium, whereas the electrical behavior is described by Maxwell's equation considering that piezoelectric media are insulating (no free volume charge):

$$div\mathbf{D} = 0. \qquad (5)$$

Equations (1)–(6) constitute a complete set of differential equations which can be solved with appropriate mechanical (displacements and forces) and electrical (potential and charge) boundary conditions. An equivalent description of above boundary value problem is Hamilton's variational principle as extended to piezoelectric media

$$\delta \int \left(L + W \right) dt = 0 \qquad (6)$$

where the operator d denotes first-order variation and the Lagrangian term *L* is determined by energies available in piezoelectric medium and *W* is the virtual work of external mechanical and electrical forces.

In the finite element method the body to be computed is subdivide into small discrete elements, the so called finite elements. The mechanical displacement u and forces f as well as the electrical potential ϕ and charge **q** are determined at the nodes of these elements. The values of these mechanical and electrical quantities at an arbitrary position on the element are given by a linear combination of polynomial interpolation function *N(x, y, z)* and the nodal point values of these quantities as coefficient. For an element with *n* nodes (nodal coordinates: (x_i, y_i, z_i); i = 1, 2 …, n) the continuous displacement function u*(x, y, z)* (vector of order three). For example can be evaluated from its discrete nodal point vectors as follows (the quantities with "^" are the nodal point values of one element):

$$\mathbf{u}(x,y,z) = \mathbf{N}_u(x,y,z)\,\hat{\mathbf{u}}(x_i,y_i,z_i) \qquad (7)$$

where $\hat{\mathbf{u}}$ is the vector of nodal point displacements and $\mathbf{N}_u$ is is the interpolation function for the displacement.

All other mechanical and electrical quantities x are similarly interpolated with appropriate interpolation functions $\mathbf{N}_x$. With the interpolation functions for the displacement ($\mathbf{N}_u$) and the electrical potential ($\mathbf{N}_\phi$), (2) and (3) can be written:

$$\begin{aligned} \mathbf{E} &= -grad\phi = grad(\mathbf{N}_\phi\hat{\phi}) = -\mathbf{B}_\phi\hat{\phi} \\ \mathbf{S} &= \mathbf{Bu} = \mathbf{B}\,\mathbf{N}_u\hat{\mathbf{u}} = \mathbf{B}_u\hat{\mathbf{u}} \end{aligned} \qquad (8)$$

The substitution of the polynomial interpolation function into (7) yields a set of linear differential equations that describe one single piezoelectric finite element.

$$\begin{aligned} \mathbf{m}\hat{\ddot{\mathbf{u}}} + \mathbf{d}_m\hat{\dot{\mathbf{u}}} + \mathbf{k}_{uu}\hat{\mathbf{u}} + \mathbf{k}_{u\phi}\hat{\phi} &= \hat{\mathbf{f}} \\ \mathbf{k}_{u\phi}{}^T\hat{\mathbf{u}} + \mathbf{k}_{\phi\phi}\hat{\phi} &= \hat{\mathbf{q}} \end{aligned} \qquad (9)$$

where $\hat{\dot{\mathbf{u}}}, \hat{\ddot{\mathbf{u}}}$ are vectors of nodal velocities, accelerations, $\mathbf{k}_{uu}$ is mechanical stiffness matrix, $\mathbf{d}_m$ is mechanical damping matrix, $\mathbf{k}_{u\phi}$ is piezoelectric coupling matrix, $\mathbf{k}_{\phi\phi}$ is dielectric stiffness matrix, $\mathbf{m}$ is mass matrix, $\mathbf{f}$ is mechanical forces and $\mathbf{q}$ is electrical charges.

Each element of the mesh is connected to its neighboring elements at the global nodes and the displacement is continuous from one element to the next. The element degrees of freedom (DOF) ($\hat{\mathbf{u}}, \hat{\phi}$) are related to the global DOF ($\mathbf{U}$, $\mathbf{\Phi}$) by the mean of the localization. The Hamilton's principle (7) must be verified for the whole structure, which results in (by summation of the contribution from each finite element).

$$\begin{vmatrix} \mathbf{M} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{U}} \\ \dot{\mathbf{U}} \end{vmatrix} + \begin{vmatrix} 0 & \mathbf{D}_m \\ 0 & 0 \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{U}} \\ \dot{\mathbf{U}} \end{vmatrix} + \begin{vmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{u\phi}^T & \mathbf{K}_{\phi\phi} \end{vmatrix} \begin{vmatrix} \mathbf{U} \\ \Phi \end{vmatrix} = \begin{vmatrix} \mathbf{F} \\ \mathbf{Q} \end{vmatrix} \qquad (10)$$

Couple the mechanical variables $\mathbf{U}$ and the electrical potentials $\mathbf{\Phi}$; $\mathbf{F}$ represents the external forces applied to the structure and $\mathbf{Q}$ the electric charges brought to the electrodes.

3 AXISYMMETRIC ANALYSIS

Note, that the shape of designing piezoceramic sensors is generally cylindrical. The requirements of computing power are decreased extremely, if task is defined like as an axisymmetric problem. For simplicity only mechanical part of problem is mentioned in detail. The problem of stress distribution in bodies of revolution (axisymmetric solids) under axisymmetric loading is of our considerable interest. The mathematical problems presented are very similar to those of plane stress and plane strain, the situation is two dimensional [2]. By symmetry, the two components of displacements in any plane section of the body along its axis of symmetry define completely the state of strain and, therefore, the state of stress. Such a crosssection is shown in fig. 1. If r and z denotes respectively the radial and axial coordinates of a point, with u and v being the corresponding displacements $\mathbf{u}$.

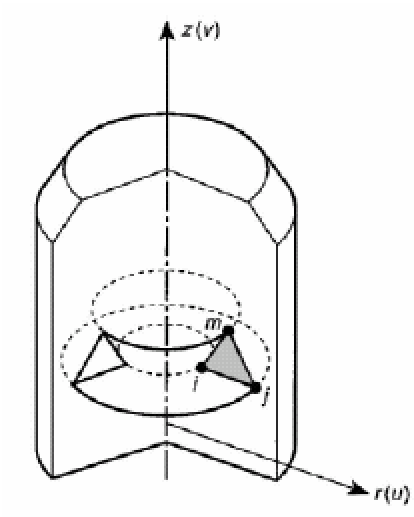

Fig. 1: *Element of axisymmetric solid*

The volume of material associated with an ‘element’ is now that of a body of revolution indicated in fig. 2., and all integrations have to be referred to this. In plane stress or strain problems it was shown [2] that internal work was associated with three strain components in the coordinate plane, the stress component normal to this plane not being involved due to zero values of either the stress or the strain.

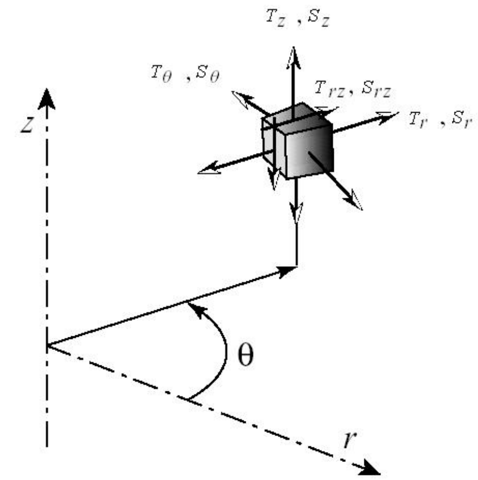

Fig. 2: *Axisymmetric strain and stresses*

In the axisymmetrical situation any radial displacement automatically induces a strain in the circumferential direction, and as the stresses in this direction are certainly non-zero, this fourth component of strain and of the associated stress has to be considered. Here lies the essential difference in the treatment of the axisymmetric situation.

As already mentioned, four components of strain have now to be considered. These are, in fact, all the non-zero strain components possible in an axisymmetric deformation. The strain vector defined below lists the strain components involved and defines them in terms of the displacements of a point. Using the displacement functions for i-node of element, then

$$\mathbf{S} = \begin{vmatrix} \mathbf{S}_r \\ \mathbf{S}_z \\ \mathbf{S}_\theta \\ \mathbf{S}_{rz} \end{vmatrix} = \begin{vmatrix} \partial u / \partial r \\ \partial v / \partial z \\ u / r \\ \partial u / \partial z + \partial v / \partial r \end{vmatrix} = \mathbf{Bu} = \mathbf{B}_u \hat{\mathbf{u}}, \qquad \mathbf{B}_{ui} = \begin{vmatrix} \partial N_{ui} / \partial r & 0 \\ 0 & \partial N_{ui} / \partial z \\ N_{ui} / r & 0 \\ \partial N_{ui} / \partial z & \partial N_{ui} / \partial r \end{vmatrix} \qquad (11)$$

For a linear hyperelastic material, and ignoring thermal and prestress effects, the most general equation consistent with axisymmetry takes the form [3]:

$$\mathbf{T} = \begin{vmatrix} \mathbf{T}_r \\ \mathbf{T}_z \\ \mathbf{T}_\theta \\ \mathbf{T}_{rz} \end{vmatrix} \begin{vmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & 0 \\ c_{14} & c_{24} & 0 & c_{44} \end{vmatrix} = \begin{vmatrix} \mathbf{S}_r \\ \mathbf{S}_z \\ \mathbf{S}_\theta \\ \mathbf{S}_{rz} \end{vmatrix} \qquad (12)$$

To retain axisymmetry, the cross-coupling between the shear strain and hoop stress must vanish. Consequently $c_{34} = c_{43} = 0$.

4 CONCLUSION

A finite-element calculation schema for 2D and 3D simulation of anisotropic piezoceramics and its modification for axisymmetric problem are presented. This finite-element method allows the solution of numerous problems encountered in piezoelectric transducer design. One of the main problems is the simultaneous appearance of various vibrational modes with quite different physical characteristics. The simulations allow a deeper understanding of the physical mechanisms of acoustic wave propagation in piezoelectric sensors.

Such simulations are used to optimize sensor design with respect to bandwidth, sensitivity and signal to noise ratio. Notes, which are mentioned in this paper, were used to develop own algorithm of finite element method. The reason is to understand noise influence on geometry of sensors and mechanical vibration influence on the electrical impedance spectrum. The algorithm is under development in MATLAB. The results are compared with ANSYS.

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