THE SYSTEM OF CLASS C WITH EIGENVALUE RESTRICTIONS

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ABSTRACT

This paper describes the simplified system belonging to class C. It´s vector field can be considered to be parted into three regions. The dynamical behavior in each region is described by the roots of two characteristic polynomials, where only one coefficient is supposed to be different. This prototype can behave chaotically as it is numerically proved. The circuitry implementation is also presented and verified via PSpice simulator.

1 INTRODUCTION

Since the chaos has been confirmed in several disciplines of common life, it becomes a topic of increasing interest of many scientists. This fact is striking in the field of electronics and communication. Simple chaos generators producing wideband signals becomes very important, mostly because it´s implementation in coders, modulators and real-time speech securing. Chaotic oscillators play a significant role also as the reference models for descibing a more complex events such as chemical reactions, fluid mixing procedures, etc.

Dynamical systems belonging to class C [1] are autonomous, deterministic and have three-segment piecewise linear (PWL) vector fields. We can express any member of this group in compact matrix form

$$\frac{d}{d\,t}\mathbf{x} = \mathbf{A}\cdot\mathbf{x} + \mathbf{b}\cdot h\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right), \qquad h\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) = \frac{1}{2}\left(\left|\,\mathbf{w}^{\mathrm{T}}\mathbf{x}+1\right| - \left|\,\mathbf{w}^{\mathrm{T}}\mathbf{x}-1\,\right|\right). \tag{1}$$

Dynamical behavior of the system (1) in both outer regions $D_{\pm 1}$ of the state space is determined by roots of the characteristic polynomial

$$\det\left(s\cdot\mathbf{E}-\mathbf{A}\right) = \left(s-\upsilon_1\right)\left(s-\upsilon_2\right)\left(s-\upsilon_3\right) = s^3 - q_1 s^2 + q_2 s - q_3 = 0\,, \tag{2a}$$

and similarly for inner region D_0

$$\det\left[\,s\cdot\mathbf{E}-\left(\mathbf{A}+\mathbf{b}\cdot\mathbf{w}^{\mathrm{T}}\right)\right] = \left(s-\mu_1\right)\left(s-\mu_2\right)\left(s-\mu_3\right) = s^3 - p_1 s^2 + p_2 s - p_3 = 0\,, \tag{2b}$$

where **E** denotes the unity matrix. For possible chaotic solution there should be one real and a pair of complex conjugated eigenvalues in each region forming an unstable spiral.

Such configurations will lead to general decomposed solution with eigenspaces $R^3 = R^2 \oplus R$

$$x(t) = C_3 \exp(\lambda_3 t)\,\vec{\xi}_3 + 2C_c \exp(\sigma t)\left[\cos(\omega t + \phi_c)\,\vec{\xi}_{real} - \sin(\omega t + \phi_c)\,\vec{\xi}_{imag}\right]. \qquad (3)$$

Dynamical system (1) has just one equilibrium point per region, in detail

$$\left(\mathbf{A} + \mathbf{b}\cdot\mathbf{w}^{\mathbf{T}}\right)\bar{\mathbf{x}}_{inner} = 0 \;\;\rightarrow\;\; \bar{\mathbf{x}}_{inner} = (0 \;\; 0 \;\; 0)^T, \qquad \bar{\mathbf{x}}_{outer} = \pm\mathbf{A}^{-1}\cdot\mathbf{b}. \qquad (4)$$

Cardan rule will help us to find a relations between set of eigenvalues and their equivalent values p_i, q_i. Substituing $s = q_1/3 + \Lambda$ into (2a) yields a reduced cubic polynomial

$$\Lambda^3 + \left(q_2 - \frac{q_1^{\,2}}{3}\right)\Lambda + \eta = 0, \qquad \eta = \frac{q_1 q_2}{3} - q_3 - \frac{2q_1^{\,3}}{27}. \qquad (5)$$

To finish this computation, assume

$$\Delta = 4\left(q_2 - \frac{q_1^{\,2}}{3}\right)^3 + 27\eta^2. \qquad (6)$$

As we expected, one real eigenvalue is

$$\lambda_3 = \frac{q_1}{3} + \Lambda_R, \qquad (7)$$

where

$$\Lambda_R = \left(-\frac{\eta}{2} + \sqrt{\frac{\eta^2}{4} + \frac{1}{27}\left(q_2 - \frac{q_1^{\,2}}{3}\right)^3}\right)^{\frac{1}{3}} + \left(-\frac{\eta}{2} - \sqrt{\frac{\eta^2}{4} + \frac{1}{27}\left(q_2 - \frac{q_1^{\,2}}{3}\right)^3}\right)^{\frac{1}{3}}, \qquad (8)$$

and a pair of complex conjugated numbers equals

$$\sigma = \frac{1}{6}\left(2q_1 - 3\Lambda_R\right), \qquad \omega = \frac{1}{2}\sqrt{4\left(q_2 - \frac{q_1^{\,2}}{3}\right) + 3\Lambda_R^{\,2}}. \qquad (9)$$

The same results last for a case of polynomial (2b) with exception $p_i \rightarrow q_i$. Now let fix a set of parameters in the outer regions on their nominal values for the purpose of producing a double-scroll attractor

$$q_1 = -0.6, \qquad q_2 = 0.846, \qquad q_3 = -1.295. \qquad (10)$$

and assume two coefficient of the characteristic polynomials (2) are the same. Mathcad program has been implemented to visualize individual cases (see fig. 1). Clearly, there are a few major restrictions for q_i and p_i values due to the natural dissipativity and nonsatisfying of Ruth-Hurwitz criterion

$$\{\, q_1 q_2 > q_3 \;\;\vee\;\; q_1 < 0 \,\} \;\wedge\; \{\, p_1 p_2 > p_3 \;\;\vee\;\; p_1 < 0 \,\}, \qquad q_3 < 0 \;\wedge\; p_3 > 0. \; (11)$$

A desired eigenvalue area is attainable only for $p_3 \neq q_3$, $p_3 \in \langle -5, 5\rangle$. Then we are able to write down a derived state equations

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = q_3 x - q_2 y + q_1 z + (p_3 - q_3)\cdot h(x). \qquad (12)$$

A better idea about eigenvalues restrictions give us a Viet´s formulas

$$2\sigma + \lambda_3 = 2\tilde{\sigma} + \tilde{\lambda}_3 = q_1, \qquad \sigma^2 + \omega^2 + 2\sigma\lambda_3 = \tilde{\sigma}^2 + \tilde{\omega}^2 + 2\tilde{\sigma}\tilde{\lambda}_3 = q_2,$$

$$\lambda_3\left(\sigma^2 + \omega^2\right) = q_3, \qquad \tilde{\lambda}_3\left(\tilde{\sigma}^2 + \tilde{\omega}^2\right) = p_3. \tag{13}$$

It is not hard to determine exact values of the last coefficient when a corresponding eigenvalues crosses the imaginary axis of complex plane (Hopf bifurcation occurs). These essential numbers are $p_1 = -1.530732861$, $p_2 = -2.158\overline{3}$ and $p_3 = 0.5076$.

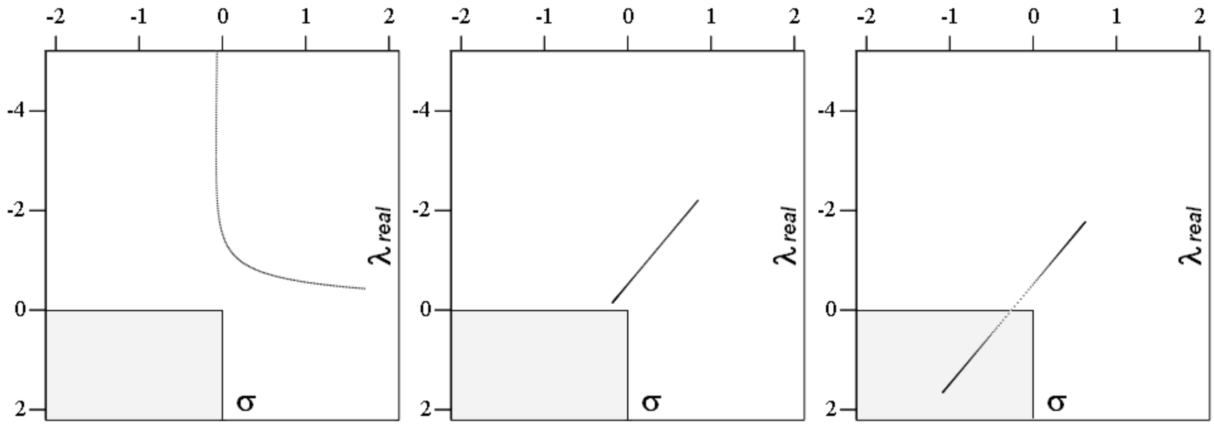

Fig. 1: *Migration of eigenvalues for* $p_1 \neq q_1$, $p_2 \neq q_2$ *and* $p_3 \neq q_3$

It could be probably usefull to verify dynamical system (12), i.e. answer a question if it has bounded solution, sensitive dependance on initial conditions and dense attractor. That is we must compute the spectrum of one-dimensional Lyapunov exponents defined as

$$LE\left(x_0, y_0 \in T_{\mathbf{x}(t)}\mathbf{R}^3\right) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\left\| D_x\Phi\left(t, \mathbf{x_0}\right)\mathbf{y_0}\right\|}{\left\|\mathbf{y_0}\right\|}, \qquad \sum_{i=1}^{3} LE_i < 0. \tag{14}$$

where brackets denote an Euclidean norm used in estimation routine and $D_x\Phi\left(t, \mathbf{x_0}\right)\mathbf{y_0}$ are variational equations associated with the original system. The essential *LE* value is the largest number which must be positive. *LE* measured in the direction of the flow should be close to zero and remaining one must be negative. A constrain in (14) suggests that volume element defined on the state trajectory is shrinking as time approaches to infinity. By performing many simulations we can also conclude that *LE* don´t depend on a choice of initial conditions or norm. Fig. 2 shows some results of numerical analysis. First picture shows a spectrum of *LE* computed repeatedly for initial conditions $(0 \quad 1 \quad 0)^T$ and time $t_{start}=0$, $t_{final}=500$. For a maximal value of *LE* also Kaplan-Yorke dimension of the attractor is established. Second picture illustrates a transient event, which must be omitted as we are waiting until a trajectory is on the attractor. Three-dimensional perspective views on time evolution of state are also presented for the cases $p_3 = 0.4$ (period four limit cycle) and $p_3 = 2.4$ (double-scroll chaotic attractor). As parameter p_3 increases a chaotic solution alternates with periodic orbits, a period-doubling bifurcation sequence has been observed. Last figure proves a sensitive dependance on tiny changes of initial state. One waveform corresponds to $(0.5 \quad 0 \quad 0)^T$ and other $(0.5001 \quad 0 \quad 0)^T$.

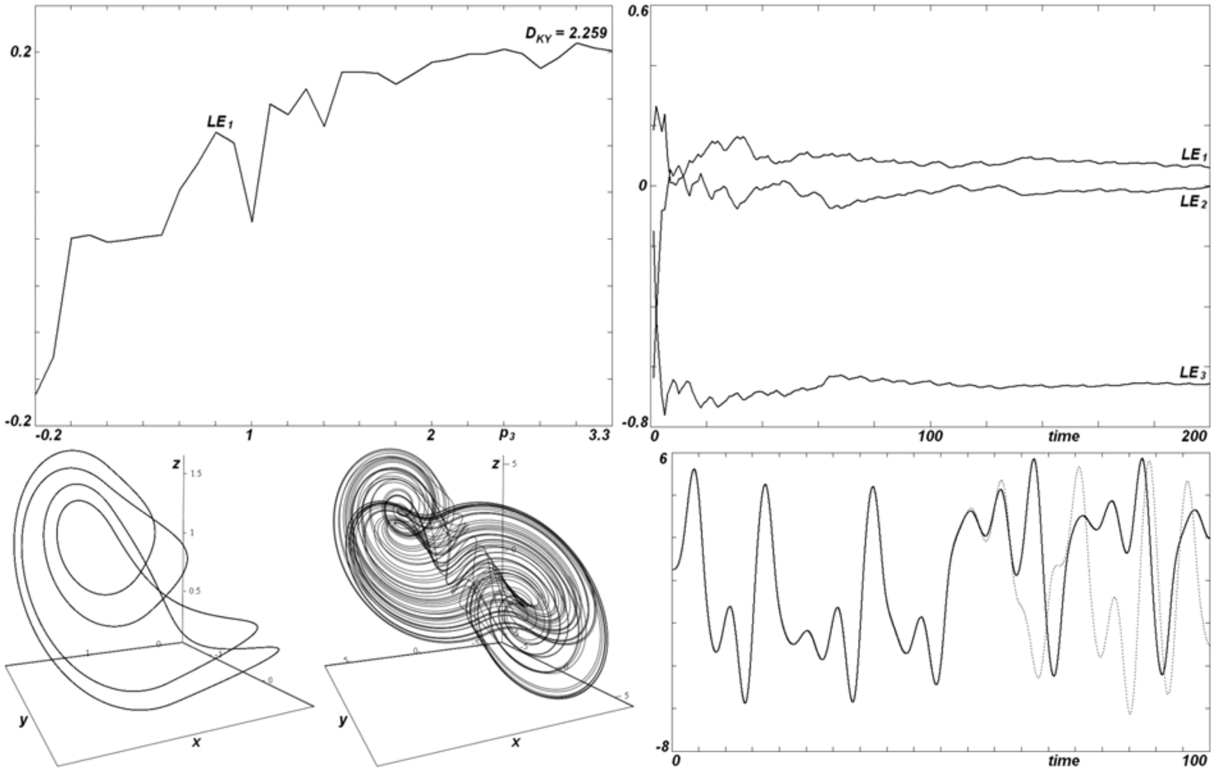

Fig. 2: *Spectrum of LE,* $LE_1 = f(p_3)$*,* $LE_i = f(0.653)$ *and typical attractors*

2 CIRCUITRY IMPLEMENTATION

The new dynamical system (12) has one big advantage if compared to a classical first ODE equivalent of Chua´s equation [1]. It is much more easier to practically realize (12) with four independent parameters than a system with six parameters. Straigtforward circuit synthesis is usually based on basic building blocks such as inverting integrator or summing amplifier. In this case, a cascade connection of integrators are coupled with multiple feedback branches and PWL transfer function. However, we can follow a completely different approach. A linear part of the vector field can be realized by a higher order admittance function, i.e.

$$Y(s)=\frac{I(s)}{U(s)}=\frac{Z_2(s)Z_4(s)}{Z_1(s)Z_3(s)Z_5(s)}=s^3-q_1\,s^2+q_2\,s-q_3\,. \qquad (15)$$

For such purpose we can use the so called Antoniou´s general impedance converter (GIC). It is a one-port network with input admittance given generally as (15). Assume two GICs

$$Y_1(s)=s^3\,C^3 R\,R_{gic}+s^2\,C^2 R_{gic}\,, \qquad Y_2(s)=s\,C\,R^{-1}R_y\,, \qquad (16)$$

connected in parallel as shown in fig. 3. To complete the oscillator, it is sufficient to construct a voltage controlled current source with desired AV characteristics. To date, several techniques has been developed [4]. Note that some segments of PWL resistor have a negative slope due to the fact that fixed points of (12) are the zeroes of the nonlinear function. That´s why a negative imittance is employed (current has inverse orientation). After the circuit´s time and current rescaling we can take $R=1\ k\Omega$, $C=100\ nF$.

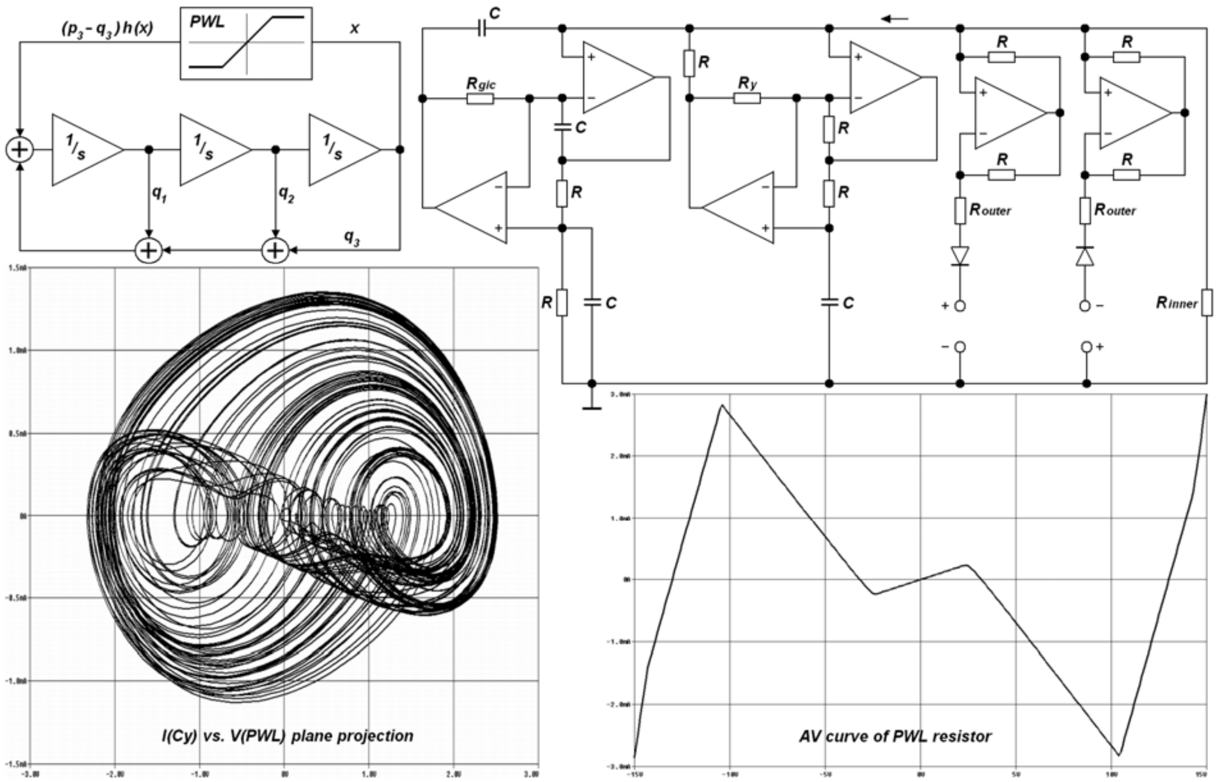

Fig. 3: *Two ways how to realize proposed chaotic oscillator, PSpice simulation*

3 CONCLUSSION

The aim of this work was to find the simplest dynamical system belonging to class C. Its practical implementation working in hybrid voltage/current mode is also presented. All parameters including a breakpoint voltages can be tuned independently and continuously as required for proper function of the circuit. We can reach a higher frequency domain by using modern and commercially available electronic devices such as CFOA, OTA, etc.

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