SOLUTION OF THE DIFFERENCE EQUATION

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ABSTRACT

The solution of the second order nonhomogeneous difference equation

$$y(n+2)-1,25y(n+1)+0,78125y(n) = \mathrm{e}^{\mathrm{j}\omega(n+2)} - \mathrm{e}^{\mathrm{j}\omega n}$$

is presented in this article. First, the equation is solved by theory of difference equations with method of variation of constants and then with $\mathcal{Z}$-transform. The results obtained are compared.

1 INTRODUCTION

This article presents the solution of the second order nonhomogeneous difference equation

$$y(n+2)-1,25y(n+1)+0,78125y(n) = \mathrm{e}^{\mathrm{j}\omega(n+2)} - \mathrm{e}^{\mathrm{j}\omega n}. \tag{1}$$

This equation is solved with the aid of two methods namely the method of variation of constants and the method of $\mathcal{Z}$-transform. Difference equation (1) can describe the characteristics of the 2nd order linear time invariant discrete system, and is used for the implementation on the microprocessors and signal processors.

2 SOLUTION WITH THE AID OF THEORY OF DIFFERENCE EQUATIONS

To solve equation (1) we use the method of variation of constants [1]. Put $n=-2$ in (1). We get

$$y(0)-1,25y(-1)+0,78125y(-2) = \mathrm{e}^{\mathrm{j}\omega 0} - \mathrm{e}^{\mathrm{j}\omega(-2)}.$$

Assuming that the system is not active for $n<0$ we have $y(-1)=y(-2)=0$, and the first initial condition $y(0)$ is

$$y(0) = 1-\mathrm{e}^{-2\mathrm{j}\omega}.$$

Substitute $n=-1$ in equation (1). We obtain

$$y(1)-1,25y(0)+0,78125y(-1)=\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}.$$

Since $y(-1)=0$ and $y(0)$ was computed above, we get the second initial condition $y(1)$:

$$y(1)=1,25y(0)+\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}=1,25(1-\mathrm{e}^{-2\mathrm{j}\omega})+\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}.$$

Thus, the two initial conditions are

$$y(0) = 1-\mathrm{e}^{-2\mathrm{j}\omega}, \tag{2}$$
$$y(1) = 1,25(1-\mathrm{e}^{-2\mathrm{j}\omega})+\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}. \tag{3}$$

Let us solve corresponding homogeneous difference equation

$$y(n+2)-1,25y(n+1)+0,78125y(n)=0 \tag{4}$$

with respect to equation (1). Roots of characteristic equation

$$\lambda^2-1,25\lambda+0,78125=0$$

are $\lambda_{1,2}=p_{1,2}=0,625\pm \mathrm{j}0,625$. Therefore homogeneous equation (4) has a pair of linear independent solutions $y_1(n)=p_1^n, y_2(n)=p_2^n$. General solution of homogeneous equation has the form

$$y(n)=C_1y_1(n)+C_2y_2(n)=C_1p_1^n+C_2p_2^n,$$

where C_1 and C_2 are arbitrary constants, and $n=0,1,2,\ldots$.
Now, we will find the particular solution of equation (1). Particular solution $y_p(n)$ can be found in the form

$$y_p(n)=u_1(n)y_1(n)+u_2(n)y_2(n),$$

where $y_1(n)$ and $y_2(n)$ form a pair of linear independent solutions of homogeneous equation (4). In our case we have

$$y_p(n)=u_1(n)p_1^n+u_2(n)p_2^n, \tag{5}$$

where $u_1(n)$, $u_2(n)$ are unknown functions, defined in accordance with recommendations in [1] by formula

$$\begin{aligned} u_1(n) &= -\sum_{r=0}^{n-1}\frac{g(r)y_2(r+1)}{W(r+1)}, \\ u_2(n) &= \sum_{r=0}^{n-1}\frac{g(r)y_1(r+1)}{W(r+1)}. \end{aligned}$$

Casoratian $W(r+1)$ has the form

$$W(r+1)=\begin{vmatrix} p_1^{r+1} & p_2^{r+1} \\ p_1^{r+2} & p_2^{r+2} \end{vmatrix}=p_1^{r+1}p_2^{r+1}(p_2-p_1).$$

Substituting in and simplifying relation (5) leads to the expression

$$y_p(n) = \frac{e^{2j\omega}-1}{p_2-p_1} \cdot \left[-\frac{e^{j\omega n}-p_1^n}{e^{j\omega}-p_1} + \frac{e^{j\omega n}-p_2^n}{e^{j\omega}-p_2} \right].$$

Then the general solution of nonhomogeneous equation has the form

$$y(n) = C_1 p_1^n + C_2 p_2^n + y_p(n),$$

where C_1 and C_2 are arbitrary constants, after expressing of $y_p(n)$

$$y(n) = C_1 p_1^n + C_2 p_2^n + \frac{e^{2j\omega}-1}{p_2-p_1} \cdot \left[-\frac{e^{j\omega n}-p_1^n}{e^{j\omega}-p_1} + \frac{e^{j\omega n}-p_2^n}{e^{j\omega}-p_2} \right].$$

Now we are able to determine constants C_1 and C_2 from initial conditions. We get

$$y(0) = C_1 + C_2 + \frac{e^{2j\omega}-1}{p_2-p_1} \cdot \left[-\frac{e^0-p_1^0}{e^{j\omega}-p_1} + \frac{e^0-p_2^0}{e^{j\omega}-p_2} \right] = C_1 + C_2,$$

$$y(1) = C_1 p_1 + C_2 p_2 + \frac{e^{2j\omega}-1}{p_2-p_1} \cdot \left[-\frac{e^{j\omega}-p_1}{e^{j\omega}-p_1} + \frac{e^{j\omega}-p_2}{e^{j\omega}-p_2} \right] = C_1 p_1 + C_2 p_2.$$

According to initial conditions (2), (3) we solve the system of two equations with two unknown variables:

$$\begin{aligned} C_1 + C_2 &= 1 - e^{-2j\omega}, \\ C_1 p_1 + C_2 p_2 &= 1{,}25(1 - e^{-2j\omega}) + e^{j\omega} - e^{-j\omega}. \end{aligned}$$

We get

$$\begin{aligned} C_1 &= \frac{1}{p_2-p_1} \left[-p_1(1-e^{-2j\omega}) - e^{j\omega} + e^{-j\omega} \right], \\ C_2 &= \frac{1}{p_2-p_1} \left[p_2(1-e^{-2j\omega}) + e^{j\omega} - e^{-j\omega} \right]. \end{aligned}$$

Finally, solution of equation (1) satisfying initial conditions is

$$\begin{aligned} y(n) = C_1 p_1^n + C_2 p_2^n + y_p(n) = \frac{p_1^n}{p_2-p_1} \cdot \left[-p_1(1-e^{-2j\omega}) - e^{j\omega} + e^{-j\omega} \right] + \\ + \frac{p_2^n}{p_2-p_1} \cdot \left[p_2(1-e^{-2j\omega}) + e^{j\omega} - e^{-j\omega} \right] + \frac{e^{2j\omega}-1}{p_2-p_1} \cdot \left[-\frac{e^{j\omega n}-p_1^n}{e^{j\omega}-p_1} + \frac{e^{j\omega n}-p_2^n}{e^{j\omega}-p_2} \right]. \end{aligned} \tag{6}$$

3 $\mathcal{Z}$-TRANSFORM

Let us recall that $\mathcal{Z}$-transform of the function $f(n+k)$ is expressed by formula

$$\mathcal{Z}\{f(n+k)\} = z^k \left[F(z) - \sum_{m=0}^{k-1} \frac{f(m)}{z^m} \right].$$

Since
$$\mathcal{Z}\{a^n\} = \frac{z}{z-a},$$
then the $\mathcal{Z}$-transform of each term in equation (1) is:

$$\begin{aligned}
\mathcal{Z}\{y(n+2)\} &= z^2\left[Y(z) - \frac{y(0)}{z^0} - \frac{y(1)}{z}\right] = z^2Y(z) - z^2y(0) - zy(1), \\
\mathcal{Z}\{y(n+1)\} &= z\left[Y(z) - \frac{y(0)}{z^0}\right] = zY(z) - zy(0), \\
\mathcal{Z}\{y(n)\} &= Y(z), \\
\mathcal{Z}\{\mathrm{e}^{\mathrm{j}\omega(n+2)}\} &= z^2\left[\frac{z}{z-\mathrm{e}^{\mathrm{j}\omega}} - \frac{\mathrm{e}^{\mathrm{j}\omega 0}}{z^0} - \frac{\mathrm{e}^{\mathrm{j}\omega}}{z}\right] = \frac{z^3}{z-\mathrm{e}^{\mathrm{j}\omega}} - z^2 - z\mathrm{e}^{\mathrm{j}\omega}, \\
\mathcal{Z}\{\mathrm{e}^{\mathrm{j}\omega n}\} &= \frac{z}{z-\mathrm{e}^{\mathrm{j}\omega}}.
\end{aligned}$$

The $\mathcal{Z}$-transform of equation (1) equals

$$z^2Y(z) - z^2y(0) - zy(1) - 1,25zY(z) + 1,25zy(0) + 0,78125Y(z) = \frac{z^3}{z-\mathrm{e}^{\mathrm{j}\omega}} - z^2 - z\mathrm{e}^{\mathrm{j}\omega} - \frac{z}{z-\mathrm{e}^{\mathrm{j}\omega}}$$

and

$$Y(z) = \frac{z^2-1}{z^2-1,25z+0,78125}\cdot\frac{z}{z-\mathrm{e}^{\mathrm{j}\omega}} + \frac{-z^2 - z\mathrm{e}^{\mathrm{j}\omega} + zy(1) + (z^2-1,25z)y(0)}{z^2-1,25z+0,78125}.$$

Moreover, we have

$$Y(z) = Y_{\mathrm{ZSR}} + Y_{\mathrm{ZIR}} = \mathcal{Z}\{y_{\mathrm{P}}(n)\} + \mathcal{Z}\{y_{\mathrm{H}}(n)\},$$

where $Y_{\mathrm{ZSR}} = \mathcal{Z}\{y_{\mathrm{P}}(n)\}$ is the Zero State Response and $Y_{\mathrm{ZIR}} = \mathcal{Z}\{y_{\mathrm{H}}(n)\}$ is the Zero Input Response of the system. Denote $Y_{\mathrm{H}}(z) = Y_{\mathrm{ZIR}}$. Then we have

$$Y_{\mathrm{H}}(z) = \frac{zy(1) + (z^2-1,25z)y(0)}{z^2-1,25z+0,78125} = \frac{zy(1) + (z^2-1,25z)y(0)}{(z-p_1)(z-p_2)}.$$

Let us find the inverse $\mathcal{Z}$-transform of $Y_{\mathrm{H}}(z)$. We use the formula

$$f(n) = \mathcal{Z}^{-1}\{F(z)\} = \frac{1}{2\pi\mathrm{j}}\oint_\Gamma F(z)z^{n-1}\mathrm{d}z = \sum_{k=1}^{K} \operatorname{res} F(z)\big|_{z=z_k},$$

where Γ is the simple, closed, positively oriented and piecewise smooth curve enclosing all the singular points $z_1, z_2, \ldots, z_k$ of integrand, and the symbol "res" means residuum of the function. Then

$$\begin{aligned}
y_{\mathrm{H}}(n) &= \frac{1}{2\pi\mathrm{j}}\oint_\Gamma \frac{\left[zy(1) + (z^2-1,25z)y(0)\right]z^{n-1}}{(z-p_1)(z-p_2)}\mathrm{d}z = \\
&= \operatorname{res} Y_{\mathrm{H}}(z)z^{n-1}|_{z=p_1} + \operatorname{res} Y_{\mathrm{H}}(z)z^{n-1}|_{z=p_2} = \\
&= \frac{1}{p_2-p_1}\left[y(1)\left(p_2^n - p_1^n\right) + y(0)\left(p_2^n(-p_1) - p_1^n(-p_2)\right)\right].
\end{aligned}$$

After the substitution of initial conditions (2), (3) we get

$$y_{\mathrm{H}}(n)=\frac{1,25(p_2^n-p_1^n)(1-\mathrm{e}^{-2\mathrm{j}\omega})+(\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega})(p_2^n-p_1^n)+(1-\mathrm{e}^{-2\mathrm{j}\omega})(-p_1p_2^n+p_1^np_2)}{p_2-p_1}=$$

$$=\frac{p_1^n}{p_2-p_1}\left[-p_1(1-\mathrm{e}^{-2\mathrm{j}\omega})-\mathrm{e}^{\mathrm{j}\omega}+\mathrm{e}^{-\mathrm{j}\omega}\right]+\frac{p_2^n}{p_2-p_1}\left[p_2(1-\mathrm{e}^{-2\mathrm{j}\omega})+\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}\right].$$

Denote $Y_{\mathrm{P}}(z)=Y_{\mathrm{ZSR}}$. Then

$$\begin{aligned} Y_{\mathrm{P}}(z) &= \frac{(z^2-1)z}{(z^2-1,25z+0,78125)(z-\mathrm{e}^{\mathrm{j}\omega})}-\frac{z^2+z\mathrm{e}^{\mathrm{j}\omega}}{z^2-1,25z+0,78125}\cdot\frac{z-\mathrm{e}^{\mathrm{j}\omega}}{z-\mathrm{e}^{\mathrm{j}\omega}}= \\ &= \frac{z(\mathrm{e}^{2\mathrm{j}\omega}-1)}{(z-\mathrm{e}^{\mathrm{j}\omega})(z-p_1)(z-p_2)}. \end{aligned}$$

We obtain the particular solution in the form

$$\begin{aligned} y_{\mathrm{P}}(n) &= \frac{1}{2\pi\mathrm{j}}\oint_{\Gamma}\frac{z(e^{2\mathrm{j}\omega}-1)z^{n-1}}{(z-\mathrm{e}^{\mathrm{j}\omega})(z-p_1)(z-p_2)}\mathrm{d}z= \\ &= \operatorname{res} Y_{\mathrm{P}}(z)z^{n-1}|_{z=\mathrm{e}^{\mathrm{j}\omega}}+\operatorname{res} Y_{\mathrm{P}}(z)z^{n-1}|_{z=p_1}+\operatorname{res} Y_{\mathrm{P}}(z)z^{n-1}|_{z=p_2} \end{aligned}$$

and after the simplification we have

$$y_{\mathrm{P}}(n)=\frac{\mathrm{e}^{2\mathrm{j}\omega}-1}{p_2-p_1}\cdot\left[-\frac{\mathrm{e}^{\mathrm{j}\omega n}-p_1^n}{\mathrm{e}^{\mathrm{j}\omega}-p_1}+\frac{\mathrm{e}^{\mathrm{j}\omega n}-p_2^n}{\mathrm{e}^{\mathrm{j}\omega}-p_2}\right].$$

The final solution via $\mathcal{Z}$-transform is equal to

$$y(n)=y_{\mathrm{H}}(n)+y_{\mathrm{P}}(n)=\frac{p_1^n}{p_2-p_1}\cdot\left[-p_1(1-\mathrm{e}^{-2\mathrm{j}\omega})-\mathrm{e}^{\mathrm{j}\omega}+\mathrm{e}^{-\mathrm{j}\omega}\right]+$$

$$+\frac{p_2^n}{p_2-p_1}\cdot\left[p_2(1-\mathrm{e}^{-2\mathrm{j}\omega})+\mathrm{e}^{\mathrm{j}\omega}-\mathrm{e}^{-\mathrm{j}\omega}\right]+\frac{\mathrm{e}^{2\mathrm{j}\omega}-1}{p_2-p_1}\cdot\left[-\frac{\mathrm{e}^{\mathrm{j}\omega n}-p_1^n}{\mathrm{e}^{\mathrm{j}\omega}-p_1}+\frac{\mathrm{e}^{\mathrm{j}\omega n}-p_2^n}{\mathrm{e}^{\mathrm{j}\omega}-p_2}\right]. \quad (7)$$

4 CONCLUSION

The same solution of difference equation (1) has been found with both methods. The $\mathcal{Z}$-transform can solve partial characteristics of the system easily, but the method of variation of constants is more universal. The speed of processors rapidly increases and we need to know the entire solution of difference equations to describe corresponding dynamical processes (transient processes treatment).

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